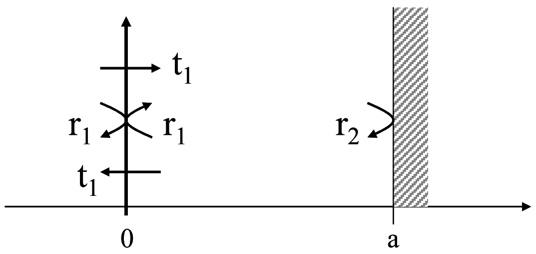
ECE/MAT 211A winter 07

Midterm solutions

1.

Solving for total net reflection @ x = 0; use scattering amplitude!

(a) $$\boxed{\left.\frac{B}{A}\right|_{x=0} = r_1 + \frac{t_1 e^{i2ka} r_2 t_1}{1 - r_1 r_2 e^{i2ka}}}$$



clearly $r_2 = -1$

$$r_1 = \frac{P_{21}}{P_{11}} = \frac{P_{12}^{\ *}}{P_{11}} = \frac{-i\kappa_0}{k + i\kappa_0} \qquad \kappa_0 = \frac{mS}{\hbar^2}$$

$$t_1 = \frac{1}{P_{11}} = \frac{k}{k + i\kappa_0} \qquad k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

$$|r_1|^2 + |t_1|^2 = \frac{k^2 + \kappa_0^{\ 2}}{k^2 + \kappa_0^{\ 2}} = 1 \qquad \text{Good!}$$

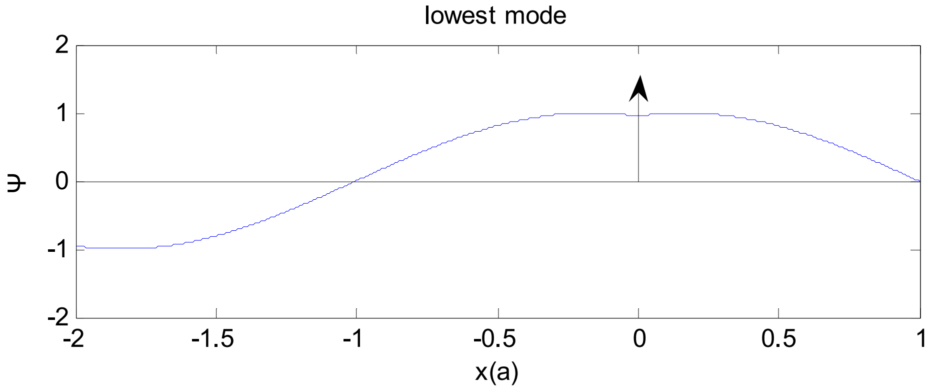
Weak barrier: $r_1 \to -i\dfrac{\kappa_0}{k}$; $t_1 \to 1$

Strong barrier: $r_1 \to -1$; $t_1 \to -i\dfrac{k}{\kappa_0}$

In all cases: $\psi'(0^+) - \psi'(0^-) = -\dfrac{2mS}{\hbar}\psi(0)$

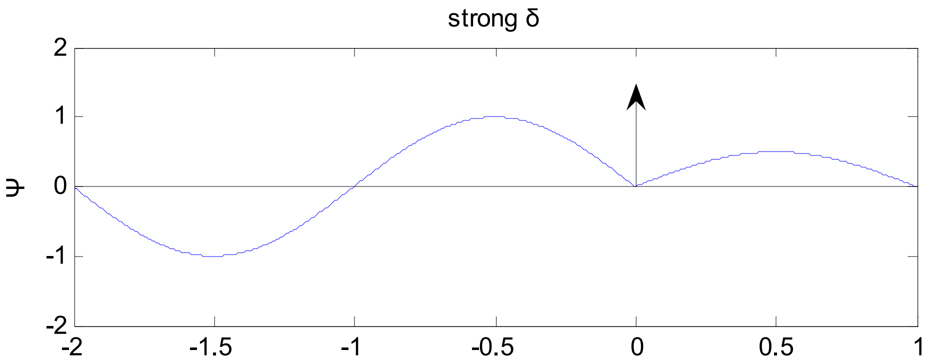
Lowest "mode" $\rightarrow$ least curvature

$\rightarrow$ no zero crossing in well



for x<0: $\psi(x) = A_1 \sin(-kx + \phi)$

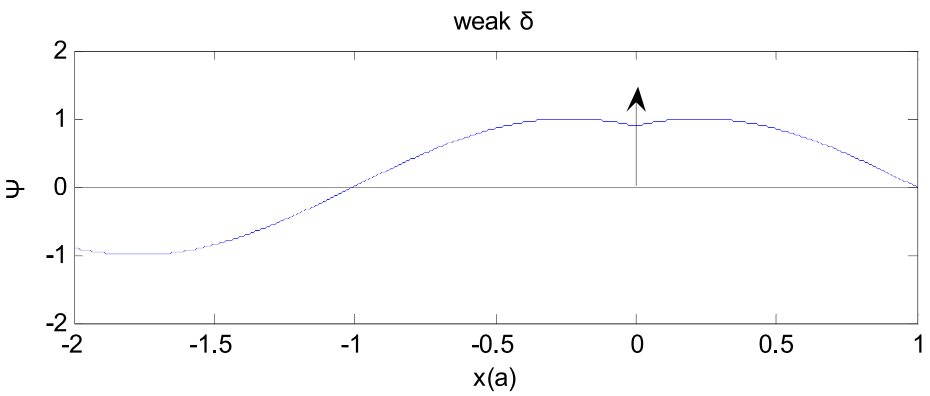
0<x<a: $\psi(x) = A_2 \sin(k(a - x))$



$\phi \rightarrow 0$

$ka \rightarrow \pi$

$r_1 \rightarrow -1$



$\phi \rightarrow \frac{\pi}{2}$

$ka \rightarrow \frac{\pi}{2}$

$r_1 \rightarrow -i\frac{\kappa_0}{k}$

(b) Clearly, $\boxed{\left|\frac{B}{A}\right| = 1}$. $\frac{B}{A} = r_1 + r_{box}$

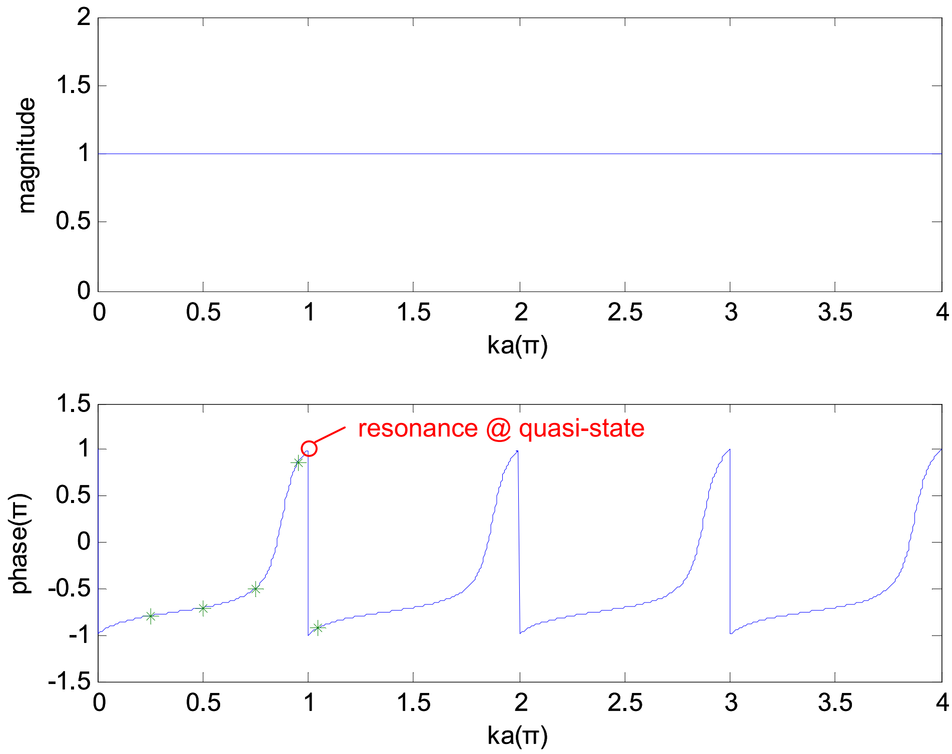
<u>Let k = κ_0 and vary "a" for phase plot</u>

$$r_1 = 0.707\angle -135^\circ$$
$$t_1 = 0.707\angle -45^\circ$$
$$\frac{B}{A} = 0.707\angle -135^\circ + \frac{0.707\angle -45^\circ \cdot 0.707\angle -45^\circ \cdot 1\angle 180^\circ \cdot 1\angle 2ka}{1 - 0.707\angle -135^\circ \cdot 1\angle 180^\circ \cdot 1\angle 2ka}$$
$$= 0.707\angle -135^\circ + \frac{0.5\angle 90^\circ \cdot 1\angle 2ka}{1 + 0.707\angle -135^\circ \cdot 1\angle 2ka}$$

Try a bunch of different ka's

ka	0.25π	0.5π	0.75π	0.95π	1.05π
$\angle \frac{B}{A}$	-0.7952π	-0.7048π	-0.5000π	0.8550π	-0.9238π

Then plot the magnitude and phase of $\frac{B}{A}$:



(c) For special case k = κ_0, $2ka = \frac{7\pi}{4}$ or $ka = \frac{7\pi}{8}$.

Thus, the sinkx function is not quite back to zero at the δ-function barrier as it would be for $ka = \pi$. More generally, quasi-states exist at resonance of the well. In this case, we observe that as minima of the denominator of the r_{box} term in $\frac{B}{A} = r_1 + r_{box}$.

This is when the 2nd term in the denominator is a negative maximum.

$$D = 1 + r_1 e^{i2ka}$$

$$\therefore \angle r_1 + 2ka = (2m+1)\pi \text{ , m = 1, 2, 3….}$$

$$r_1 = \frac{-i\kappa_0 k - \kappa_0^2}{k^2 + \kappa_0^2} \quad \text{(a third quadrant angle)}$$

$$\left(\pi + Arc\tan\frac{k}{\kappa_0}\right) + 2ka = (2m+1)\pi \quad \text{or}$$

$$\boxed{2ka = 2m\pi - Arc\tan\frac{k}{\kappa_0}} \quad \text{similar to a finite well}$$

which agrees with our special case above $Arc\tan\frac{k}{\kappa_0} = \frac{\pi}{4}$

$$k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}, \quad \kappa_0 = \frac{mS}{\hbar^2}$$

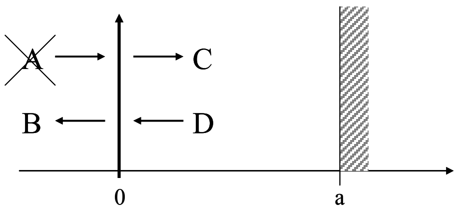
So the expression for the energies of these states is

$$\boxed{\sqrt{\frac{2m\varepsilon_m}{\hbar^2}}a = m\pi - \frac{1}{2}Arc\tan\left[\frac{2m\varepsilon_m}{\hbar^2\kappa_0^2}\right]}$$

(d) See part (a)

(e) When $A \to 0$ @ t =0

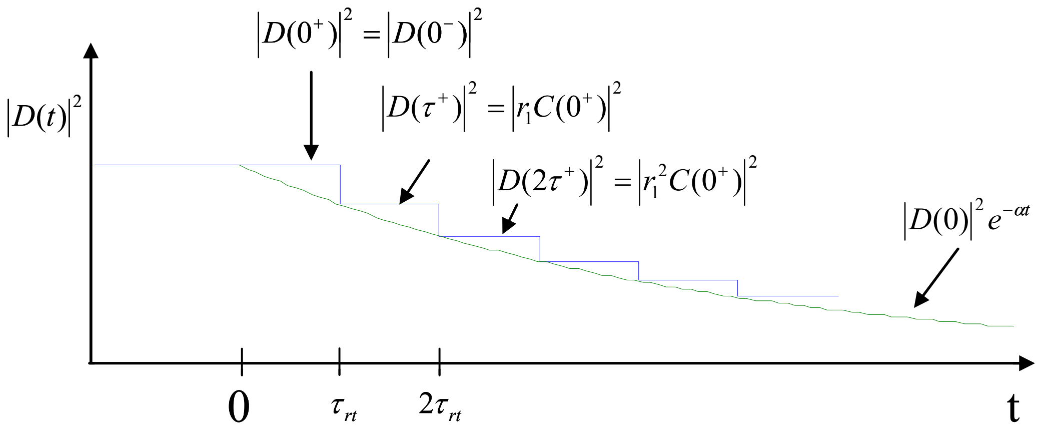
then $|B(t)|^2 = |B(0)|^2 e^{-\alpha t}$



At state where denominator of r_{box} is minimum

- roundtrip phase = 2mπ
- all recirculations in phase

Once $A \to 0$, $B = t_1 D$; $C^+ = r_1 D^-$, $D^+ = -C^- e^{i2ka}$ as always.



$\therefore \exp(-\alpha \tau_{rt}) = |r_1^2|$, $\qquad \tau_{rt} = \dfrac{2a}{v_g} = \dfrac{2am}{\hbar k}$

$$\boxed{\alpha = \frac{-\ln|r_1^2|}{\tau_{rt}} = \frac{-\hbar k}{2am}\ln\left(\frac{\kappa_0^2}{k^2+\kappa_0^2}\right) = \frac{-\hbar k}{2am}\ln\left(\frac{mS^2}{2\varepsilon\hbar^2+mS^2}\right) = \frac{\hbar k}{2am}\ln\left(1+\frac{2\varepsilon\hbar^2}{mS^2}\right)}$$

If $S \to \infty$ (strong barrier), $\alpha \to 0$ (no leakage).

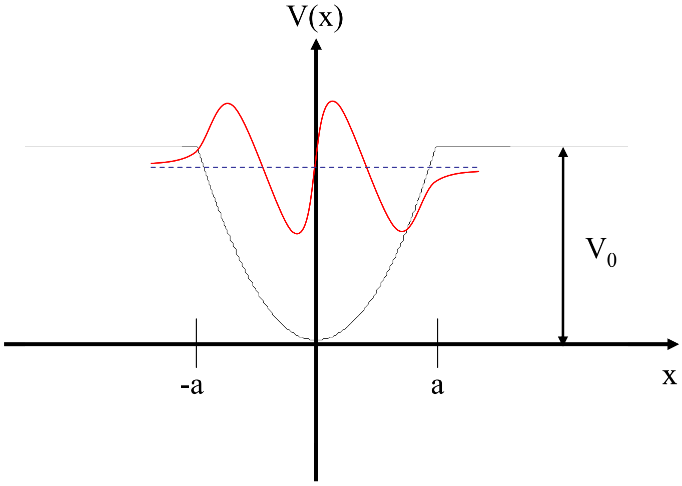
If $S \to 0$ (weak barrier), $\alpha \to \infty$ (no confinement).

Make sense!

2.

(a) They go down somewhat as the barrier is reduced for |x|>a. The upper levels go down more than the lower ones.

(b) The decay of the wavefunction is now longer and the wavefunction in general becomes a little wider as it is less confined. Inflections occur at the point where V(x) = ε_4 as always.



(c) Now we move the walls in till the ε_4 moves up to where it was originally in the unmodified well $E = \frac{7}{2}\hbar w$. Since k(x) is now the same for |x|<a, as for the unmodified well, the wavefunction is also the same for |x|<a. But @ |x| = a, the wavafunction is horizontal, because this is the cutoff value of a. In other words, $\psi_3'(Q) = 0$ (the fourth wavefunction) @ |x| =a.

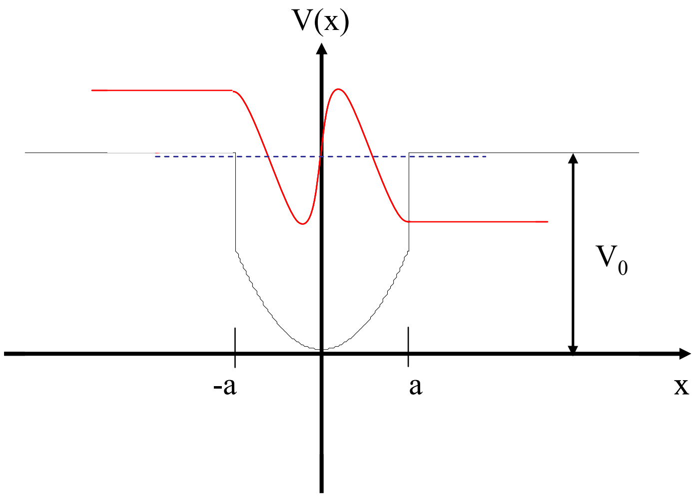
$$\psi_3 \propto H_3(Q)\cdot e^{-\frac{Q^2}{2}} = \left(8Q^3 - 12Q\right)\cdot e^{-\frac{Q^2}{2}}, \quad Q = \frac{x}{L}$$

$$\psi_3' \propto \left(H_3'(Q_a) - Q_a \cdot H_3(Q_a)\right)\cdot e^{-\frac{Q_a^2}{2}} = \left(-8Q_a^4 + 36Q_a^2 - 12\right)\cdot e^{-\frac{Q_a^2}{2}} = 0$$

$$Q_a^2 = \frac{1}{4}\left(9 + \sqrt{57}\right) = \begin{cases} 4.1375 \\ 0.3625 \end{cases}, \quad Q_a = \begin{cases} 2.0341 \\ 0.6021 \end{cases}$$. The upper root is the 2nd maximum for the 4th state.

$$\boxed{a = Q_a L = 2.034\left(\frac{\hbar}{mw}\right)^{1/2}}$$

(d) See sketch – same wavefunction as for unmodified well, extreme at |x| = a, and horizontal lines for |x| > a.



(e) The smaller root in part (c) is for the 2nd state. Again, if the eigen energy $= V_0 = \frac{7}{2}\hbar w$, then the wavefunction will be that of the unmodified harmonic oscillator for |x| < a of the 4th state, even if this happens to be the 2nd state after a is reduced.

$$\boxed{a = Q_a L = 0.6021\left(\frac{\hbar}{mw}\right)^{1/2}}$$

$$Q = \frac{x}{L}$$

$$L = \left(\frac{\hbar}{mw}\right)^{1/2}$$

Exercise on page 175

From (6·3-2a) we know:

$$V(x) = V_{00} + \frac{1}{2} M w^2 (x - x_0)^2 \tag{1}$$

$$V'(x) = M w^2 (x - x_0) \tag{2}$$

$V''(x) = Mw^2$ is a constant for all x (3)

so $\boxed{w = \sqrt{\frac{V''(a)}{M}}}$. (4)

Replacing (3) into (2) yields

$V'(a) = V''(a)(a - x_0)$

so $\boxed{x_0 = a - \frac{V'(a)}{V''(a)}}$. (5)

From (1) we know:

$$\boxed{V_{00} = V(a) - \frac{1}{2} M w^2 (a - x_0)^2 = \varepsilon - \frac{1}{2} M \frac{V''(a)}{M} (a - a + \frac{V'(a)}{V''(a)})^2 = \varepsilon - \frac{V'(a)^2}{2V''(a)}}. \tag{6}$$

since $V(a) = \varepsilon$.

From (6·3-2b) we know:

$$\varepsilon = V_{00} + \hbar w (n + \frac{1}{2})$$

$$\boxed{n = \frac{\varepsilon - V_{00}}{\hbar w} - \frac{1}{2} = \frac{\varepsilon - \varepsilon + \frac{V'(a)^2}{2V''(a)}}{\hbar \sqrt{\frac{V''(a)}{M}}} - \frac{1}{2} = \frac{1}{2}\left(\frac{V'(a)^2}{\hbar} \sqrt{\frac{M}{V''(a)^3}} - 1 \right)}. \tag{7}$$