ECE215A/MAT206A, Solid-State Engineering, Winter 2009

HW#2 Solutions

1. (a) η_{yy} and η_{zz} are identically zero from the definition of "clamped"

   (b) To derive the stress terms, we first write the form of the stiffness matrix from the clues given for a cubic solid:

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}$$

   By inspection, $P_{xx} = C_{11}\eta_{xx} + C_{12}\eta_{yy} + C_{12} + \eta_{zz} = C_{11}\eta_{xx}$, $P_{zz} = C_{12}\eta_{xx} + C_{12}\eta_{yy} + C_{11}\eta_{zz} = C_{12}\eta_{xx}$.

   (c) Assuming $\eta_{xx} = 1\times10^{-2}$, $C_{11} = 1.66\times10^{11}\ N/m^2$, $C_{12} = 0.64\times10^{11}\ N/m^2$ we get

   $P_{xx} = 1.6\times10^{9}\ N/m^2$, $P_{zz} = 0.64\times10^{9}\ N/m^2$

   (d) If replaced by an isotropic homogeneous solid, we still have 12 nonzero stiffness matrix elements. But there are now only two independent ones.

   (e) if clamps are removed, $P_{xx} = Y\eta_{xx}$ where Y is Young's modulus. $\eta_{zz} = \eta_{yy} = \sigma\, \eta_{xx} = \sigma\, P_{xx}/Y$.

2. (a) Given the following form of the compliance matrix,

0.7664	-0.2130	-0.2130	0	0	0
-0.2130	0.7664	-0.2130	0	0	0
-0.2130	-0.2130	0.7664	0	0	0
0	0	0	1.2563	0	0
0	0	0	0	1.2563	0
0	0	0	0	0	1.2563

   we expect the inverse (stiffness) matrix to have exactly the same form; i.e., all nonzero elements in the upper-left quadrant; all zeroes in the lower left and upper right quadrants; and nonzero elements only along the diagonal of the lower-right quadrant.

   (b) substitution of this matrix into Matlab and using the inv.m function (matrix inversion) leads to the result.

1.6600	0.6390	0.6390	0	0	0
0.6390	1.6600	0.6390	0	0	0
0.6390	0.6390	1.6600	0	0	0
0	0	0	0.7960	0	0
0	0	0	0	0.7960	0
0	0	0	0	0	0.7960

   Assuming the units are 10^{11} N/m^2 , we have C_{11} = 166 GPa, C_{12} = 63.9 GPa, and C_{44} = 79.6 Gpa - the three distinct coefficient for crystalline silicon !

4. (a) In the text the general form of the compliance matrix for an isotropic solid is found:

$$\begin{bmatrix}\eta_{xx}\\\eta_{yy}\\\eta_{zz}\\\eta_{yz}\\\eta_{xz}\\\eta_{xy}\end{bmatrix}=\begin{bmatrix}1/Y & -\sigma/Y & -\sigma/Y & 0 & 0 & 0\\ -\sigma/Y & 1/Y & -\sigma/Y & 0 & 0 & 0\\ -\sigma/Y & -\sigma/Y & 1/Y & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\sigma)/Y & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\sigma)/Y & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\sigma)/Y\end{bmatrix}\bullet\begin{bmatrix}P_{xx}\\P_{yy}\\P_{zz}\\P_{yz}\\P_{xz}\\P_{xy}\end{bmatrix}$$

We expect the stiffness matrix to have the form:

$$\begin{bmatrix}P_1\\P_2\\P_3\\P_4\\P_5\\P_6\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{44}\end{bmatrix}\bullet\begin{bmatrix}\eta_1\\\eta_2\\\eta_3\\\eta_4\\\eta_5\\\eta_6\end{bmatrix}$$

By definition, the stiffness matrix is the inverse of the compliance matrix. Stated mathematically, this means that if X is the matrix product of the two matrices, then $X_{MN}=\sum_K C_{MK}S_{KN}=\delta_{M,N}$ where $\delta_{M,N}$ is the Kronecker delta function (= 1 if M = N; = 0 if M $\neq$ N). It is easy to show $X_{11} = X_{22} = X_{33} = 1 = (C_{11} - 2C_{12}\sigma)/Y$ (*)_, and $X_{12} = X_{13} = X_{21} = X_{23} = X_{31} = X_{32} = 0 = [C_{12} - \sigma(C_{11}+C_{12})]/Y$ (**) . Multiplying (*) by σ and adding to (**) (to eliminate dependence on C_{11}) , we get $\sigma Y = -2C_{12}\sigma^2 + C_{12}(1 - \sigma)$, or

$$C_{12} = \sigma Y/(1 - \sigma - 2\sigma^2) .$$

Substitution of this back yields $C_{11} = Y + 2\sigma^2 Y/(1 - \sigma - 2\sigma^2)$, or

$$C_{11} = Y(1 - \sigma)/ /(1 - \sigma - 2\sigma^2)$$

To get C_{22}, we note $X_{44} = X_{55} = X_{66} = 1 = C_{44}\, 2(1+\sigma)/Y$, or

$$C_{44} = Y/[2(1 + \sigma)]$$

(b) In the notes an alternative form of the stiffness matrix is derived for an isotropic solid in terms of Lame coefficients

$$\begin{bmatrix}P_{xx}\\P_{yy}\\P_{zz}\\P_{yz}\\P_{xz}\\P_{xy}\end{bmatrix}=\begin{bmatrix}2\mu_L+\lambda_L & \lambda_L & \lambda_L & 0 & 0 & 0\\ \lambda_L & 2\mu_L+\lambda & \lambda_L & 0 & 0 & 0\\ \lambda_L & \lambda_L & 2\mu_L+\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_L & 0 & 0\\ 0 & 0 & 0 & 0 & \mu_L & 0\\ 0 & 0 & 0 & 0 & 0 & \mu_L\end{bmatrix}\bullet\begin{bmatrix}\eta_{xx}\\\eta_{yy}\\\eta_{zz}\\\eta_{yz}\\\eta_{xz}\\\eta_{xy}\end{bmatrix}$$

These can be related directly to the Young's modulus and Poisson's ratio through the stiffness coefficients. By inspection, we have

$$\lambda_L = C_{12} = \sigma Y/(1 - \sigma - 2\sigma^2)$$
$$\mu_L = Y/[2(1 + \sigma)]$$

As a self consistency check we take the sum $2\mu_L + \lambda_L = Y(1 - \sigma)/ /(1 - \sigma - 2\sigma^2) = C_{11}$, as expected.

6. The generic equation to be solved is

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 F}{\partial t^2} = 0$$

(a). Letting u = x +vt , w = x –vt , we get

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial w}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial w}\right) = \frac{\partial}{\partial u}\frac{\partial F}{\partial x} + \frac{\partial}{\partial w}\frac{\partial F}{\partial x}$$

$$= \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w \partial u} + \frac{\partial^2 F}{\partial w^2} = \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w^2}$$

(b)

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial t} = v\frac{\partial F}{\partial u} - v\frac{\partial F}{\partial w}$$

$$\frac{\partial^2 F}{\partial t^2} = \mathrm{v}\frac{\partial}{\partial u}\frac{\partial F}{\partial t} - \mathrm{v}\frac{\partial}{\partial w}\frac{\partial F}{\partial t} = \mathrm{v}^2\frac{\partial^2 F}{\partial u^2} - 2\mathrm{v}^2\frac{\partial^2 F}{\partial u \partial w} + \mathrm{v}^2\frac{\partial^2 F}{\partial w^2}$$

(c) Given the results of (a) and (b), the new wave equation is

$$\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w^2} - \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} - \frac{\partial^2 F}{\partial w^2} = 4\frac{\partial^2 F}{\partial u \partial w} = 0$$

(d) The solution is obtained by integrating with respect to u and w

$\int 4\frac{\partial^2 F}{\partial u \partial w}dw = C_1 G(u) + C_2$, where C_1 and C_2 are constants, G an *arbitrary* function.

So we can write $\int 4\frac{\partial^2 F}{\partial u \partial w}dwdu = C_1 H(u) + C_4 + C_3 I(w)$, where C_2, C_3, C_4 are constants, and H(u) and I(w) are *arbitrary* functions. Hence, the general solution is:

$$F = C_2H(x+vt) + C_3I(x-vt) + C_4 .$$

5. AlN film is “clamped” in the x-y plane and subjected to a uniform external electric field of E_0 = 10 KV/cm (=10^6 V/m) along the z axis.

(a). By definition, clamped in x and y means that there can be no strain in these axes and the only nonzero strain term will be η_3. So the connection equation will be the inverse piezoelectric relation $P_J = C_{IJ}\, \eta_J - e_{Ij}\, E_j$. To use this, we need the 6x3 form of the piezoelectric stress which can be derived from the more commonly used 3x6 form as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We also need the stiffness coefficients and values of the piezoelectric stress coefficients, which are both tabulated here:

AlN Stiffness Coefficients ($\times 10^{10}$ Newton/m^2)						
C11	**C12**	**C13**	**C14**	**C33**	**C44**	**C66**
41	14	10		39	12	13.5

	Piezoelectric Constants (Coulomb/m^2)						ε_r
Material	**e11**	**e14**	**e15**	**e22**	**e31**	**e33**	ε_r
AlN			-0.48		-0.58	1.55	8.5

The stiffness coefficients go into the general stiffness matrix for solids having hexagonal symmetry:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ 0 & C_{11} & C_{13} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

where $C_{66} = (1/2)(C_{11} - C_{12})$. Now using matrix multiplication, we can write

$$P_1 = C_{13}\,\eta_3 - e_{31}\,E_0$$

$$P_2 = C_{13}\,\eta_3 - e_{31}\,E_0$$

$$P_3 = C_{33}\,\eta_3 - e_{33}\,E_0$$

Where the other possible P components: P_4, P_5, and P_6 are necessarily zero because we assumed up front that the only nonzero strain component was η_3.

Substitution of the appropriate values leads to

$$P_1 = 10\times10^{10}\,\eta_3 + 0.58\times10^{6} \qquad [N/m^2]$$

$$P_2 = 10\times10^{10}\,\eta_3 + 0.58\times10^{6} \qquad [N/m^2]$$

$$P_3 = 39\times10^{10}\,\eta_3 - 1.55\times10^{6} \qquad [N/m^2]$$

(b) To get the surface charge density in the two interface planes, we start with the direct piezoelectric relation: $D_i = \varepsilon_{ij}E_j + e_{iJ}\,\eta_J$. When expressed in terms of the known Ez and η_3, and using the above 3x6 form of the piezoelectric stiffness matrix, we get $D_Z = \varepsilon_r\varepsilon_0\, E_0 + e_{33}\,\eta_3$. Then from electrostatics, we know $P_Z = D_Z - \varepsilon_0\, E_0 = (\varepsilon_r\text{-}1)\varepsilon_0\, E_0 + e_{33}\eta_3. = 6.6\text{x}10^{-5} + 1.55\,\eta_3$ [Cb/m^2]. Until we know the strain, no further evaluation can be done.

(c) If the AlN film is now used as a MEMS cantilever by unclamping it, by definition all the stress coefficients will be zero. So we can write from the piezoelectric inverse relation,

$P_J = 0 = C_{IJ}\,\eta_J - e_{Ij}\,E_j$, which in full matrix form becomes

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_z \end{bmatrix}$$

And when written out using the stress coefficients and $E_0 = 10^6$ V/m becomes

$$C_{11}\,\eta_1 + C_{12}\,\eta_2 + C_{13}\,\eta_3 = e_{31}\,E_Z = -0.58\text{x}10^6$$
$$C_{12}\,\eta_1 + C_{11}\,\eta_2 + C_{13}\,\eta_3 = e_{31}\,E_Z = -0.58\text{x}10^6$$
$$C_{13}\,\eta_1 + C_{13}\,\eta_2 + C_{33}\,\eta_3 = e_{33}\,E_Z = 1.55\text{x}10^6$$

This has a simple 3x3 numerical matrix form: $10^{10} \cdot \begin{bmatrix} 41 & 14 & 10 \\ 14 & 41 & 10 \\ 10 & 10 & 39 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} -0.58x10^6 \\ -0.58x10^6 \\ 1.55x10^6 \end{bmatrix}$

(d) One can use MATLAB to invert the 3x3 numerical matrix to get

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = 10^{-10} \cdot \begin{bmatrix} 0.0285 & -0.0085 & -0.0051 \\ -0.0085 & 0.0285 & -0.0051 \\ -0.0051 & -0.0051 & 0.0283 \end{bmatrix} \begin{bmatrix} -0.58x10^6 \\ -0.58x10^6 \\ 1.55x10^6 \end{bmatrix}$$

or $\eta_1 = -1.96\text{x}10^{-6}$, $\eta_2 = -1.96\text{x}10^{-6}$, and $\eta_3 = 4.98\text{x}10^{-6}$