HW#2 Solutions

1. (a) e_{yy} and e_{zz} are identically zero from the definition of "clamped"
   (b) To derive the stress terms, we first write the form of the stiffness matrix from the clues given for a cubic solid:

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}$$

   By inspection, $P_{xx} = C_{11}e_{xx} + C_{12}e_{yy} + C_{12} + e_{zz} = C_{11}e_{xx}$, $P_{zz} = C_{12}e_{xx} + C_{12}e_{yy} + C_{11}e_{zz} = C_{12}e_{xx}$.
   (c) Assuming $e_{xx} = 1\times10^{-2}$, $C_{11} = 1.66\times10^{11}\ N/m^2$, $C_{12} = 0.64\times10^{11}\ N/m^2$ we get $P_{xx} = 1.6\times10^{9}\ N/m^2$, $P_{zz} = 0.64\times10^{9}\ N/m^2$
   (d) If replaced by an isotropic homogeneous solid, we still have 12 nonzero stiffness matrix elements. But there are now only two independent ones.
   (e) if clamps are removed, $P_{xx} = Ye_{xx}$ where Y is Young's modulus. $e_{zz} = e_{yy} = \sigma\, e_{xx} = \sigma\, P_{xx}/Y$.

2. (a) Given the following form of the compliance matrix,

0.7664	-0.2130	-0.2130	0	0	0
-0.2130	0.7664	-0.2130	0	0	0
-0.2130	-0.2130	0.7664	0	0	0
0	0	0	1.2563	0	0
0	0	0	0	1.2563	0
0	0	0	0	0	1.2563

   we expect the inverse (stiffness) matrix to have exactly the same form; i.e., all nonzero elements in the upper-left quadrant; all zeroes in the lower left and upper right quadrants; and nonzero elements only along the diagonal of the lower-right quadrant.
   (b) substitution of this matrix into Matlab and using the inv.m function (matrix inversion) leads to the result.

1.6600	0.6390	0.6390	0	0	0
0.6390	1.6600	0.6390	0	0	0
0.6390	0.6390	1.6600	0	0	0
0	0	0	0.7960	0	0
0	0	0	0	0.7960	0
0	0	0	0	0	0.7960

   Assuming the units are 10^{11} N/m^2, we have C_{11} = 166 GPa, C_{12} = 63.9 GPa, and C_{44} = 79.6 GPa - the three distinct coefficient for crystalline silicon !

3. (a) P_{xx} = stress component = +10 N/Area = $+10/10^{-4} = +10^5$ N/m^2; the positive sign because the force is tensile. By definition $e_{xx} = P_{xx}/Y$ along the **x** axis. So the strain is given by $e_{xx} = 10^5/1.3\text{x}10^{11}$ N/m^2 = $7.7\text{x}10^{-7}$.
   (b) Both lateral strains are given by $e_{yy} = e_{zz} = \sigma\, P_{xx}/Y = \sigma\, e_{xx} = 0.28\cdot 7.7\text{x}10^{-7} = 2.2\text{x}10^{-7}$.

4. (a) In the text the general form of the compliance matrix for an isotropic solid is found:

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{xz} \\ e_{xy} \end{bmatrix} = \begin{bmatrix} 1/Y & -\sigma/Y & -\sigma/Y & 0 & 0 & 0 \\ -\sigma/Y & 1/Y & -\sigma/Y & 0 & 0 & 0 \\ -\sigma/Y & -\sigma/Y & 1/Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\sigma)/Y & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\sigma)/Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\sigma)/Y \end{bmatrix} \bullet \begin{bmatrix} P_{xx} \\ P_{yy} \\ P_{zz} \\ P_{yz} \\ P_{xz} \\ P_{xy} \end{bmatrix}$$

We expect the stiffness matrix to have the form:

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \bullet \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

By definition, the stiffness matrix is the inverse of the compliance matrix. Stated mathematically, this means that if X is the matrix product of the two matrices, then $X_{MN} = \sum_K C_{MK} S_{KN} = \delta_{M,N}$ where $\delta_{M,N}$ is the Kronecker delta function (= 1 if M = N; = 0 if M ≠ N). It is easy to show $X_{11} = X_{22} = X_{33} = 1 = (C_{11} - 2C_{12}\sigma)/Y$ (*)_, and $X_{12} = X_{13} = X_{21} = X_{23} = X_{31} = X_{32} = 0 = [C_{12} - \sigma(C_{11}+C_{12})]/Y$ (**) . Multiplying (*) by σ and adding to (**) (to eliminate dependence on C_{11}) , we get $\sigma Y = -2C_{12}\,\sigma^2 + C_{12}(1 - \sigma)$, or

$$C_{12} = \sigma Y/(1 - \sigma - 2\sigma^2) .$$

Substitution of this back yields $C_{11} = Y + 2\sigma^2 Y/(1 - \sigma - 2\sigma^2)$, or

$$C_{11} = Y(1 - \sigma)/ /(1 - \sigma - 2\sigma^2)$$

To get C_{22}, we note $X_{44} = X_{55} = X_{66} = 1 = C_{44}\, 2(1+\sigma)/Y$, or

(b) In the notes an alternative form of the stiffness matrix is derived for an isotropic solid in terms of Lame coefficients

$$\begin{bmatrix} P_{xx} \\ P_{yy} \\ P_{zz} \\ P_{yz} \\ P_{xz} \\ P_{xy} \end{bmatrix} = \begin{bmatrix} 2\mu_L + \lambda_L & \lambda_L & \lambda_L & 0 & 0 & 0 \\ \lambda_L & 2\mu_L + \lambda & \lambda_L & 0 & 0 & 0 \\ \lambda_L & \lambda_L & 2\mu_L + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_L & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_L & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_L \end{bmatrix} \bullet \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{xz} \\ e_{xy} \end{bmatrix}$$

These can be related directly to the Young's modulus and Poisson's ratio through the stiffness coefficients. By inspection, we have

$$\lambda_L = C_{12} = \sigma Y/(1 - \sigma - 2\sigma^2)$$

$$\mu_L = Y/[2(1 + \sigma)]$$

As a self consistency check we take the sum $2\mu_L + \lambda_L = Y(1 - \sigma)/ /(1 - \sigma - 2\sigma^2) = C_{11}$, as expected.

5. (a) For purified aluminum Y = 73 GPa and $\sigma = 0.33 \Rightarrow \lambda_L = 53.3$ GPa and $\mu_L = 27.4$ GPa.

For amorphous SiO_2 , Y = 85 GPa, and σ = 0.25 $\Rightarrow$ λ_L = 34.0 GPa and μ_L = 34.0 GPa (equal !)
(b) From the Lame coefficients for aluminum, $C_{12} = \lambda_L$ = 53 GHz, $C_{11} = 2\mu_L + \lambda_L$ = 108 GPa and $C_{44} = \mu_L$ = 27 GPa. From the Lame coefficients for amorphous SiO_2, C_{12} = 34 GPa, C_{11} = 102 GPa and C_{44} = 34 GPa.

6. The generic equation to be solved is

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0$$

(a). Letting u = x +vt , w = x –vt , we get

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial w}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial w}\right) = \frac{\partial}{\partial u}\frac{\partial F}{\partial x} + \frac{\partial}{\partial w}\frac{\partial F}{\partial x}$$

$$= \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w \partial u} + \frac{\partial^2 F}{\partial w^2} = \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w^2}$$

(b)

$$\frac{\partial^2 F}{\partial t^2} = v\frac{\partial}{\partial u}\frac{\partial F}{\partial t} - v\frac{\partial}{\partial w}\frac{\partial F}{\partial t} = v^2\frac{\partial^2 F}{\partial u^2} - 2v^2\frac{\partial^2 F}{\partial u \partial w} + v^2\frac{\partial^2 F}{\partial w^2}$$

(c) Given the results of (a) and (b), the new wave equation is

$$\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} + \frac{\partial^2 F}{\partial w^2} - \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial w} - \frac{\partial^2 F}{\partial w^2} = 4\frac{\partial^2 F}{\partial u \partial w} = 0$$

(d) The solution is obtained by integrating with respect to u and w

$\int 4\frac{\partial^2 F}{\partial u \partial w}dw = C_1 G(u) + C_2$, where C_1 and C_2 are constants, G an *arbitrary* function.

So we can write $\int 4\frac{\partial^2 F}{\partial u \partial w}dwdu = C_1 H(u) + C_4 + C_3 I(w)$, where C_2, C_3, C_4 are constants, and H(u) and I(w) are *arbitrary* functions. Hence, the general solution is:

$$F = C_2 H(x+vt) + C_3 I(x-vt) + C_4 \ .$$

7. (a). along the **x**, **y**, or **z** directions in an isotropic solid, we can use the same formulation as for a cubic solid and solve the component wave equations from the text. With a longitudinal wave $\Delta r_x = \Delta r_0 \exp[j(kx - \omega t)]$ along the x axis, for example. The solution leads to the dispersion relation $\omega = k\, c = k\, (C_{11}/\rho)^{1/2} = k\, [(2\mu_L + \lambda_L)/\rho]^{1/2}$ were c is the speed of longitudinal sound in terms of the stiffness and Lame coefficients.
(b). For the shear wave, we solve for $r_y = \Delta r_0 \exp[j(kx - \omega t)]$ or
$r_z = \Delta r_0 \exp[j(kx - \omega t)]$ and find $\omega = k\, c = k\, (C_{44}/\rho)^{1/2}$.

(c) Along the **x + y + z** axis (cube diagonal in Cartesian coordinates), we let

$u = \Delta r_x = \upsilon = \Delta r_y = z = \Delta r_z$ then the elastic wave equation becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = C_{11} \frac{\partial^2 u}{\partial x^2} + C_{44}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \left(C_{12}+C_{44}\right)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z}\right)$$

(only dependent variable is u !). We seek the solution $u = u_o e^{\left[j(k(x+y+z)/\sqrt{3} - \omega t)\right]}$ and

substitute to get

$$-\rho\omega^2 u_o = -C_{11}\frac{k^2}{3}u_o - C_{44}\left(\frac{k^2}{3}+\frac{k^2}{3}\right)u_o - \left(C_{12}+C_{44}\right)\left(\frac{k^2}{3}+\frac{k^2}{3}\right)u_o$$

$$\rho\omega^2 = C_{11}\frac{k^2}{3} + 4C_{44}\frac{k^2}{3} + 2C_{12}\frac{k^2}{3}$$

or $$\frac{\omega}{k} = \upsilon_s = \left[\frac{1}{\rho}\left(\frac{C_{11}}{3}+\frac{2C_{12}}{3}+\frac{4C_{44}}{3}\right)\right]^{1/2} = \frac{\omega}{k} = \upsilon_s = \left[\frac{1}{\rho}\left(\frac{C_{11}}{3}+\frac{2C_{12}}{3}+\frac{4C_{44}}{3}\right)\right]^{1/2}$$

(d). To get the velocities we first need the densities: a little searching reveals ρ = 2700 KG/m^3 (Al) and ρ =2600 KG/m^3 (SiO_2), so that for compressional wave along **x, y, or z**:
$c = k(C_{11}/\rho)^{1/2} = [(2\mu_L + \lambda_L)/\rho]^{1/2}$ = 6325 m/s for aluminum and 6263 m/s for SiO_2

For the shear wave along **x, y, or z**: we get $c = k(C_{44}/\rho)^{1/2} = [\mu_L/\rho]^{1/2}$ = 3162 m/s for Al and

3616 m/s for SiO_2 . For the compressional wave along **x + y + z** direction, we get

$$\frac{\omega}{k} = \upsilon_s = \left[\frac{1}{\rho}\left(\frac{C_{11}}{3}+\frac{2C_{12}}{3}+\frac{4C_{44}}{3}\right)\right]^{1/2} = \left[\frac{1}{\rho}\left(\frac{2\mu_L+\lambda_L}{3}+\frac{2\lambda_L}{3}+\frac{4\mu_L}{3}\right)\right]^{1/2} = \left[\frac{2\mu_L+\lambda_L}{\rho}\right]^{1/2}$$

So the longitudinal wave is the same along the **x + y + z** as along **x, y, or z**. This can be generalized to a compressional (or shear) wave velocity independent of orientation, just what one expects in an "isotropic" solid !

8. Clamped AlN film is "clamped" in the x-y plane and subjected to a uniform external electric field of E_0 = 10 KV/cm (=10^6 V/m) along the z axis.

(a). By definition, clamped in x and y means that there can be no strain in these axes and the only nonzero strain term will be η_3. So the connection equation will be the inverse piezoelectric relation $P_J = C_{IJ}\, \eta_J - e_{Ij}\, E_j$. To use this, we need the 6x3 form of the piezoelectric stress which can be derived from the more commonly used 3x6 form as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We also need the stiffness coefficients and values of the piezoelectric stress coefficients, which are both tabulated here:

AlN Stiffness Coefficients (x10^{10} Newton/m^2)						
C11	**C12**	**C13**	**C14**	**C33**	**C44**	**C66**
41	14	10		39	12	13.5

	Piezoelectric Constants (Coulomb/m^2)						$\boldsymbol{\varepsilon_r}$
Material	**e11**	**e14**	**e15**	**e22**	**e31**	**e33**	$\boldsymbol{\varepsilon_r}$
AlN			-0.48		-0.58	1.55	8.5

The stiffness coefficients go into the general stiffness matrix for solids having hexagonal symmetry:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ 0 & C_{11} & C_{13} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

where $C_{66} = (1/2)(C_{11} - C_{12})$. Now using matrix multiplication, we can write

$$P_1 = C_{13}\, \eta_3 - e_{31}\, E_0$$

$$P_2 = C_{13}\, \eta_3 - e_{31}\, E_0$$

$$P_3 = C_{33}\, \eta_3 - e_{33}\, E_0$$

Where the other possible P components: P_4, P_5, and P_6 are necessarily zero because we assumed up front that the only nonzero strain component was η_3. Substitution of the appropriate values leads to

$$P_3 = 39\text{x}10^{10}\ \eta_3 - 1.55\ \text{x}10^{6} \qquad [\text{N/m}^2]$$

(b) To get the surface charge density in the two interface planes, we start with the direct piezoelectric relation: $D_i = \varepsilon_{ij}E_j + e_{iJ}\ \eta_J$. When expressed in terms of the known Ez and η_3, and using the above 3x6 form of the piezoelectric stiffness matrix, we get $D_Z = \varepsilon_r\varepsilon_0\ E_0 + e_{33}\ \eta_3$. Then from electrostatics, we know $P_Z = D_Z - \varepsilon_0\ E_0 = (\varepsilon_r\text{-}1)\varepsilon_0\ E_0 + e_{33}\eta_3 = 6.6\text{x}10^{-5} + 1.55\ \eta_3$ [Cb/m^2]. Until we know the strain, no further evaluation can be done.

(c) If the AlN film is now used as a MEMS cantilever by unclamping it, by definition all the stress coefficients will be zero. So we can write from the piezoelectric inverse relation, $P_J = 0 = C_{IJ}\ \eta_J - e_{Ij}\ E_j$, which in full matrix form becomes

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_z \end{bmatrix}$$

And when written out using the stress coefficients and $E_0 = 10^6$ V/m becomes

$$C_{11}\ \eta_1 + C_{12}\ \eta_2 + C_{13}\ \eta_3 = e_{31}\ E_Z = -0.58\text{x}10^{6}$$
$$C_{12}\ \eta_1 + C_{11}\ \eta_2 + C_{13}\ \eta_3 = e_{31}\ E_Z = -0.58\text{x}10^{6}$$
$$C_{13}\ \eta_1 + C_{13}\ \eta_2 + C_{33}\ \eta_3 = e_{33}\ E_Z = 1.55\text{x}10^{6}$$

This has a simple 3x3 numerical matrix form: $10^{10}\cdot\begin{bmatrix} 41 & 14 & 10 \\ 14 & 41 & 10 \\ 10 & 10 & 39 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} -0.58x10^6 \\ -0.58x10^6 \\ 1.55x10^6 \end{bmatrix}$

(d) One can use MATLAB to invert the 3x3 numerical matrix to get

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = 10^{-10}\cdot\begin{bmatrix} 0.0285 & -0.0085 & -0.0051 \\ -0.0085 & 0.0285 & -0.0051 \\ -0.0051 & -0.0051 & 0.0283 \end{bmatrix}\begin{bmatrix} -0.58x10^6 \\ -0.58x10^6 \\ 1.55x10^6 \end{bmatrix}$$

or $\eta_1 = -1.96\text{x}10^{-6}$, $\eta_2 = -1.96\text{x}10^{-6}$, and $\eta_3 = 4.98\text{x}10^{-6}$