HW#4 Solutions, Chapter 9, Problems 6-8

6. Energy dependent scattering time of generic form: $\tau(U) = AU^{-S}$. From the text

$$<\tau>=\frac{<v^2\tau(v)>}{<v^2>}=\frac{m}{3k_BT}\frac{\int v^2\tau(v)f_o(v)v^2dv}{\int f_o(v)v^2dv}=\frac{2}{3k_BT}\frac{\int U^{3/2}\tau(U)f_o(U)dU}{\int U^{1/2}f_o(U)dU}$$

where $f_o(u)=n(\frac{h^2}{2\pi mk_BT})^{3/2}e^{-U/k_BT}$ (low-density particle distribution function)

So
$$<\tau>=\frac{2}{3k_BT}\int_0^\infty\frac{u^{3/2}Au^{-s}e^{-U/k_BT}dU}{\int_0^\infty u^{1/2}e^{-U/k_BT}dU}$$

$$U'=U/k_BT\Rightarrow dU=k_BTdU'$$

$$<\tau>=\frac{2}{3k_BT}\frac{(k_BT)^{5/2}(k_BT)^{-s}\int_0^\infty AU^{3/2-S}e^{-U'}dU'}{(k_BT)^{3/2}\int_0^\infty(U')^{1/2}e^{-U'}dU'}$$

Now recall definition $\Gamma(z)=\int_0^\infty e^{-x}x^{z-1}dx$. Thus by expansion of the Gamma function

$$\int_0^\infty(U')^{1/2}e^{-U'}du'=\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{1}{2})=\frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty(U')^{3/2-S}e^{-U'}du'=\Gamma(\frac{5}{2}-S)=(\frac{3}{2}-S)\Gamma(\frac{3}{2}-s)$$

So
$$<\tau>=\frac{2A}{3k_BT}(k_BT)^{1-S}\frac{\Gamma(5/2-S)}{\sqrt{\pi}/2}=\frac{4A}{3\sqrt{\pi}(k_BT)^S}\Gamma(5/2-S)$$

$$=\frac{4A}{3\sqrt{\pi}(k_BT)^S}(\frac{3}{2}-S)\Gamma(\frac{3}{2}-S)$$

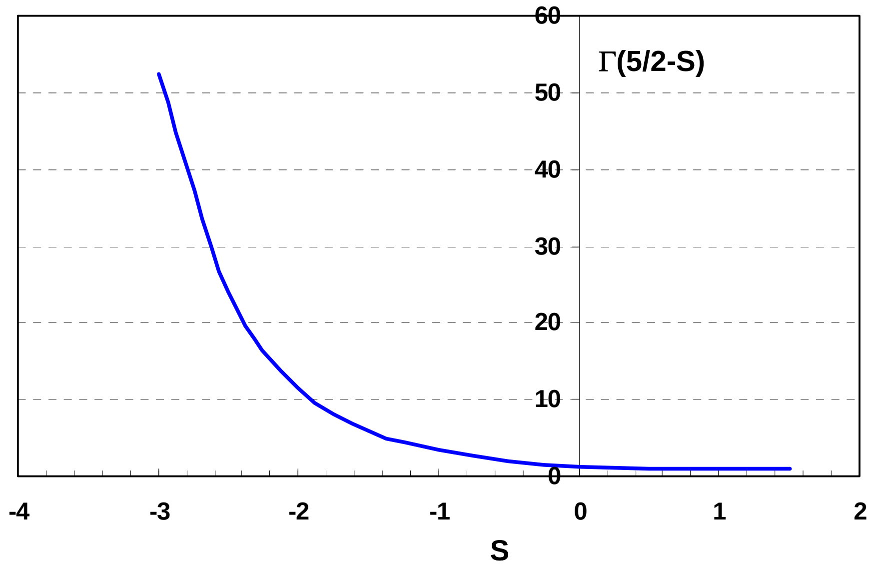

Figure for Problem 6.

Note: the gamma function is not as mysterious as it might first appear. In fact, it included in Microsoft Excel as a worksheet function gammaln(x), that is mathematically equal to ln[Γ(x)]. From applied mathematics, we know that Γ(n) has a special definition when n is an integer in terms of the factorial function, Γ(n) = (n-1)! This is seen in the Figure, for example at S = -2.5, Γ[5/2-(-5/2)] = Γ(5) = (5-1)! = 24.

7. Boltzmann transport in a uniform electric field

$$\frac{df}{dt} = -\frac{\partial f}{\partial \vec{v}}\frac{d\vec{v}}{dt} - \frac{f - f_0}{\tau} = 0 \text{ in the steady state}$$

$$\Rightarrow f - f_o = -q\frac{\vec{E}_o \tau}{m} \cdot \frac{\partial f}{\partial \vec{v}} = -\frac{q\tau \vec{E}_o}{m} \cdot \vec{\nabla}_v f$$

As in the notes we approximate $\vec{\nabla}_v f \cong \vec{\nabla}_v f_o$

So $$f \simeq f_o - \frac{q\tau \vec{E}_o}{m} \cdot \vec{\nabla}_v f_o \equiv f_o - \vec{v}_o \cdot \vec{\nabla} v_v f_o$$

once we define $\vec{v}_o \equiv q\tau \vec{E}_o / m$

Now recall form of Taylor's series for increment, h

$$P(x+h) = P(x) + hP'(x) + \ldots$$

By associating $x \rightarrow \vec{v}, h \rightarrow -\vec{v}_o, P \rightarrow f_o$

We find $$f \cong f_o(\vec{v}-\vec{v}_o) = Ae^{-\frac{1}{2}m(\vec{v}-\vec{v}_o)^2 / k_B T}$$

The validity of this expression rests on accuracy of the Taylor expansion

$$f \simeq f_o - \frac{q\tau\vec{E}_o}{m}\cdot\vec{\nabla}f_o$$

$$\vec{\nabla}f_o = \frac{mv}{k_B T} f_o \Rightarrow f \simeq f_o - \frac{q\tau E_o}{m}\frac{mv}{k_B T} f_o$$

$$f = f_o(1 - v_o \frac{mv}{k_B T})$$

Validity of Taylor expansion requires that f is small deviation from f_0, which implies

$$f_o \Rightarrow v_o \frac{mv}{k_B T} \ll 1$$

or $$v_o \frac{mv}{k_B T} \approx \frac{mv_o^2}{k_B T} < 1 \Rightarrow v_o \ll \sqrt{\frac{k_B T}{m}}$$

8. Electrical conductivity for energy-independent mean-free-path

Define mean-free-path, $\lambda = v\cdot\tau$; Apply Maxwell distribution

$$\sigma = \frac{ne^2 <\tau>}{m} \quad ; <\tau> = \frac{<\tau v^2>}{<v^2>} = \frac{\lambda <v>}{<v^2>}$$

$$f_o = Ae^{-(1/2)mv^2/k_B T}$$

$$<v> = \int_0^\infty Ave^{-(1/2)mv^2/k_B T} v^2 dv = \int_0^\infty Av^3 e^{-\alpha v^2} dv$$

$$<v> = \frac{A}{2\alpha^2} = \frac{2A(k_B T)^2}{m^2}$$

$$= \frac{4}{8}3(k_B T/m)^2\sqrt{\frac{\pi k_B T}{m/2}}A = \frac{3\sqrt{2\pi}}{2}(k_B T/m)^{5/2}A = 3\sqrt{\frac{\pi}{2}}(k_B T/m)^{5/2}\cdot A$$

So $$<\tau> = \frac{\lambda <v>}{<v^2>} = \frac{2\lambda A(k_B T)^2/m^2}{(3A\sqrt{2\pi}/2)(k_B T/m)^{5/2}} = \lambda\frac{4}{3}\sqrt{\frac{m}{2\pi k_B T}}$$

$$<\sigma> \equiv \frac{ne^2 <\tau>}{m} = \frac{4}{3}\frac{\lambda n e^2}{\sqrt{2\pi m k_B T}}$$

Alternatively, could do integrals using Γ function

Define $$U = \frac{1}{2} m v^2 / k_B T \qquad dU = \frac{mv}{k_B T} dv$$

$$dU = \frac{m}{k_B T}\sqrt{\frac{2Uk_B T}{m}} dv$$

$$dU = \sqrt{\frac{2mU}{k_B T}} dv$$

$$< v^2 >= \int_0^\infty A(2Uk_B T/m)^2 e^{-U} \sqrt{\frac{k_B T}{2mU}} dU$$

$$= \int_0^\infty A(k_B T/m)^{5/2} (2U)^{3/2} e^{-U} dU$$

$$\int_0^\infty U^{3/2} e^{-U} dU = \Gamma(5/2) = \Gamma(3/2+1) = 3/2 \cdot \Gamma(3/2)$$

$$= (3/2)\Gamma(1/2+1) = 3/4\Gamma(1/2) = 3\sqrt{\pi}/4$$

So $$< v^2 >= A(k_B T/m)^{5/2} 2^{3/2} \cdot 3\sqrt{\pi}/4 = A(k_B T/m)^{5/2} \cdot 3\sqrt{\pi/2}$$

as derived with Gaussian integrals above.