HW#5 Solutions, Chapter 10, Problems 1-4

Problem 1. Semiclassical correction to kinetic theory

(a) N-type phosphorous doped Si $1\times10^{18}/cm^3$, $\delta = 0.02\Omega - cm$

We know $\sigma = ne\mu$ so, $\rho \equiv 1/\sigma = 1/(ne\mu) \Rightarrow \mu = 1/(ne\rho)$. In practical units,

$\mu = \left(10^{18}\right)\left(1.6e^{-19}\right)\left(0.02\right) = 312$ cm^2/(V-s) or in MKS units $3.12\text{x}10^{-2}$ m^2/(V-s)

(b) $\mu = e\tau / m^* \Rightarrow \tau = m^* \mu / e$. The correct effective mass to use for the scattering time is the conductivity mass $m_e / m_c^* = (1/6)(2/0.98 + 4/0.19) \Rightarrow m_c^* = 0.26 m_e$ since scattering involves dynamic mechanical effects. Substitution of MKSA values yields $\tau = 46\ fs$.

(c) Einstein relation $D = k_B T \mu / e = 8.06\times10^{-4} m^2/s$.

Problem 2. Cyclotron resonance in spheroidal band

We start with the semiclassical dynamic equation, $\hbar \frac{d\vec{k}}{dt} = \frac{e}{\hbar}\frac{\partial U}{\partial \vec{k}} \times \vec{B}$. From the given spheroidal form for U(k), $\frac{\partial U}{\partial k_x} = \frac{\hbar^2 k_x}{m_T}, \frac{\partial U}{\partial k_y} = \frac{\hbar^2 k_y}{m_T}, \frac{\partial U}{\partial k_z} = \frac{\hbar^2 k_z}{m_L}$. So if we confine *B* to the *x*, *y* plane $\vec{B} = B_x\hat{x} + B_y\hat{y}$ the semi-classical equation becomes

$$\frac{dk_x}{dt} = \frac{-ek_z B_y}{m_L} \qquad \frac{dk_y}{dt} = \frac{ek_z B_x}{m_L} \qquad \frac{dk_z}{dt} = e\left(\frac{k_x B_y}{m_T} - \frac{k_y B_x}{m_T}\right) \qquad (1)$$

Note: $$\frac{1}{\hbar^2}\frac{\partial U}{\partial \vec{k}} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{k_x}{m_T} & \frac{k_y}{m_T} & \frac{k_z}{m_L} \\ B_x & B_y & 0 \end{vmatrix} = \hat{x}\left(\frac{-k_z B_y}{m_L}\right) + \hat{y}\left(\frac{k_z B_x}{m_L}\right) + \hat{z}\left(\frac{k_x B_y}{m_T} - \frac{k_y B_x}{m_T}\right)$$

We seek oscillatory solutions in *k* space (would be oscillatory in real space too, but do not need that for the present problem). Hence,

$$\vec{k} = \vec{k}_0 e^{j\omega t};\ \frac{dk_x}{dt} = j\omega k_{0x};\ \frac{dk_y}{dt} = j\omega k_{0y};\ \frac{dk_z}{dt} = j\omega k_{0z}$$

and (1) becomes

$$0=-j\omega k_{0x}-\frac{eB_y}{m_L}k_{0z} \qquad 0=-j\omega k_{0y}+\frac{eB_x}{m_L}k_{0z}$$

$$0=-j\omega k_{0z}+\frac{eB_y}{m_T}k_{0x}-\frac{eB_x}{m_T}k_{0y}$$

We can write this in elegant matrix form as

$$0=\begin{pmatrix} -j\omega & 0 & \mp\frac{eB_y}{m_L} \\ 0 & -j\omega & \pm\frac{eB_x}{m_L} \\ \pm\frac{eB_y}{m_T} & \mp\frac{eB_x}{m_T} & -j\omega \end{pmatrix}\begin{bmatrix} k_{0x} \\ k_{oy} \\ k_{0z} \end{bmatrix}$$

From linear algebra we know this matrix equation has non-trivial solutions for the column vector $\vec{k}$ if and only if the matrix is singular, i.e., the determinant vanishes. So

$$Det[\;]=-j\omega\left(-\omega^2+\frac{e^2B_x^2}{m_Tm_L}\right)\mp\frac{eB_y}{m_L}\left(\pm j\omega\frac{eB_y}{m_T}\right)=-j\omega\left(-\omega^2+\frac{e^2B_x^2}{m_Tm_L}+\frac{e^2B_y^2}{m_Tm_L}\right)=0$$

The non-trivial solution comes from inside the parenthesis.

$$\Rightarrow \omega^2=\frac{e^2B_x^2}{m_Tm_L}+\frac{e^2B_y^2}{m_Tm_L}=\frac{e^2\left|\vec{B}\right|^2}{m_Tm_L}$$

or
$$\omega=\frac{e\left|\vec{B}\right|}{\sqrt{m_Tm_L}},\left|\vec{B}\right|=\sqrt{B_x^2+B_y^2}$$

Problem 3. Semiclassical transport in germanium

(a) For any energy band $\vec{v}_g=(\hbar)^{-1}\vec{\nabla}U\left(\vec{k}\right)\equiv v_{gx}\hat{x}+v_{gy}\hat{y}+v_{gz}\hat{z}$ and $d\vec{r}/dt=\vec{v}_g$, where $\vec{r}$ defines the location of the wave-packet center. For the specific energy band $U=(\hbar^2/2)(\alpha k_x^2+\alpha k_y^2+\alpha k_z^2+2\beta k_xk_y+2\beta k_xk_z+2\beta k_yk_z)$, we get

$\vec{v}_{gx}=\hbar\left(\alpha k_x+\beta(k_y+k_z)\right)$; $\vec{v}_{gy}=\hbar\left(\alpha k_y+\beta(k_x+k_z)\right)$; $\vec{v}_{gz}=\hbar\left(\alpha k_z+\beta(k_x+k_y)\right)$, and thus $dx/dt=\hbar\left(\alpha k_x+\beta(k_y+k_z)\right)$; $dy/dt=\hbar\left(\alpha k_y+\beta(k_x+k_z)\right)$;

$dz/dt=\hbar\left(\alpha k_z+\beta(k_x+k_y)\right)$,

(b) The vector semiclassical equation of motion is $\hbar d\vec{k}/dt = q\left(\vec{E} + \hbar^{-1}\vec{\nabla}_k U \times \vec{B}\right) - \hbar\vec{k}/\tau$.

For $\vec{E} = E_0\hat{x}$ and $\vec{B} = B_0\hat{z}$, the component equations become

$$x : \hbar dk_x / dt = q\left[E_0 + B_0[\alpha k_y + \beta(k_x + k_z)]\right] - \hbar k_x / \tau$$

$y : \hbar dk_y / dt = B_0[\alpha k_x + \beta(k_y + k_z)] - \hbar k_y / \tau$ and $z : \hbar dk_z / dt = -\hbar k_z / \tau$.

In the steady state, $dk_x/dt = 0$, so if $B_0 = 0$, the x component has obvious solution $k_x = qE_0\tau/\hbar$

(c) From (a), the motion of the wavepacket along x is $dx/dt = \hbar\left(\alpha k_x + \beta(k_y + k_z)\right)$.

Assuming B = 0, the only nonzero ballistic terms from (b) are for k_x (ballistic means collisionless, so τ is set to infinity): $k_x = qE_0 t/\hbar + C$ where C is a constant that must be zero if particle starts at rest (rest means zero velocity and hence $k_x = 0$) . Substitution yields $dx/dt = \hbar\alpha(qE_0 t/\hbar)$ or $x = \hbar\alpha(qE_0 t^2/2\hbar) + D$, where D is a second constant. The distance traveled in real space between t = 0 and t = τ is just $\Delta x = x(t = \tau) - x(t = 0) = \hbar\alpha(qE_0\tau^2/2\hbar)$

Problem 4. Hall Effect with two carrier types and spherical bands

Semi-classical equations:

<u>Electrons:</u> $$\hbar\frac{d\vec{k}_e}{dt} = -e\left(E + \frac{1}{\hbar}\vec{\nabla}_k U_e\left(\vec{k}\right)\times\vec{B}\right) - \frac{\hbar\vec{k}_e}{\tau_e} = -e\left(\vec{E} + \hbar\frac{\vec{k}_e}{m_e^*}\times\vec{B}\right) - \frac{\hbar\vec{k}_e}{\tau_e}$$

<u>Holes:</u> $$\hbar\frac{d\vec{k}_h}{dt} = e\left(\vec{E} + \frac{1}{\hbar}\vec{\nabla}_k U_h\left(\vec{k}\right)\times\vec{B}\right) - \frac{\hbar\vec{k}_h}{\tau_h} = e\left(\vec{E} + \frac{\hbar\vec{k}_h\times\vec{B}}{m_h^*}\right) - \frac{\hbar\vec{k}_h}{\tau_h}$$

In Hall configuration, $\vec{B} = B_0\hat{z},\ \vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}, \Rightarrow$

$$\vec{k}_e \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_{ex} & k_{ey} & k_{ez} \\ 0 & 0 & B_0 \end{vmatrix} = \hat{x}k_{ey}B_0 - \hat{y}k_{ex}B_0;\quad \vec{k}_h \times \vec{B} = \hat{x}k_{hy}B_0 - \hat{y}k_{hx}B_0$$

In a steady state all time derivatives go to zero, so semi-classical equations become:

Electrons | **Holes**

$$\begin{aligned} 0 &= -eE_x - \frac{e\hbar k_{ey} B_0}{m_e^*} - \frac{\hbar k_{ex}}{\tau_e} \\ 0 &= -eE_y + \frac{e\hbar k_{ex} B_0}{m_e^*} - \frac{\hbar k_{ey}}{\tau_e} \\ 0 &= -eE_z - \frac{\hbar k_{ez}}{\tau_e} \end{aligned} \tag{1}$$

$$\begin{aligned} 0 &= eE_x + \frac{e\hbar k_{hy} B_0}{m_h^*} - \frac{\hbar k_{hx}}{\tau_h} \\ 0 &= eE_y - \frac{e\hbar k_{hx} B_0}{m_h^*} - \frac{\hbar k_{hy}}{\tau_h} \\ 0 &= eE_z - \frac{\hbar k_{hz}}{\tau_h} \end{aligned} \tag{2}$$

These is just six equations in six unknowns, $k_{ex}, k_{ey}, k_{ez}, k_{hx}, k_{hy}, k_{hz}$. To solve, multiply the first of set (1) by $eB_0\tau_e / m_e^*$ and add to the second of set (1). Then multiply the first of set (2) by $-eB_0\tau_h / m_h^*$ and add to the second of set (2). We get

$$0 = \frac{-e^2 B_0 \tau_e E_x}{m_e^*} - \frac{e^2 \hbar k_{ey}}{\left(m_e^*\right)^2} - eE_y - \frac{\hbar k_{ey}}{\tau_e}$$

$$0 = \frac{-e^2 E_x B_0 \tau_h}{m_h^*} - \frac{-e^2 \hbar k_{hy} B_0^2 \tau_h}{\left(m_h^*\right)^2} + eE_y - \frac{\hbar k_{hy}}{\tau_h}$$

These can be rewritten as:

$$k_{ey}\left[\frac{-e^2 \hbar B_0^2 \tau_e}{\left(m_e^*\right)^2} - \frac{\hbar}{\tau_e}\right] = eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*} \qquad \text{and} \qquad k_{hy}\left[\frac{-e^2 B_0^2 \tau_h}{\left(m_h^*\right)^2} - \frac{\hbar}{\tau_h}\right] = -eE_y + \frac{e^2 E_x B_0 \tau_h}{m_h^*}$$

Ignoring terms in B_0^2, we get

$$k_{ey} \approx \frac{-\tau_e}{\hbar}\left(eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*}\right) \qquad\qquad k_{hy} \approx \frac{\tau_h}{\hbar}\left(eE_y - \frac{e^2 B_0 \tau_h E_x}{m_h^*}\right)$$

Recall that the electron current in a given band is defined in the semiclassical model as

$$J_{ey} = -eE_y \int_{band} \frac{\hbar k_{ey}}{m_e^*} f_e d\vec{k}_e = \frac{e\hbar}{m_e^*}\frac{\tau_e}{\hbar}\left(eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*}\right)\underbrace{\int f_e d^3k}_{n}$$

Similarly, the hole electrical current is

$$J_{hy} = eE_y \int_{band} \frac{\hbar k_h}{m_h^*} f_h d\vec{k}_h = \frac{e\hbar}{m_h^*}\frac{\tau_h}{\hbar}\left(eE_y - \frac{e^2 B_0 \tau_h E_x}{m_h^*}\right)\underbrace{\int f_h d^3k}_{p}$$

Since the net current along y axis is zero, we can write $J_{ey} + J_{hy} = 0$ or

$$0 = \frac{e^2\tau_e}{m_e^*}\left(E_y + \frac{eB_0\tau_e E_x}{m_e^*}\right) n + \frac{e^2\tau_h}{m_h}\left(E_y - \frac{eB_0\tau_h E_x}{m_h^*}\right) p$$

Solving for E_y we get

$$\left(ne\mu_e + pe\mu_h\right) E_y = \left[\frac{ne^3\tau_e}{m_e^*}\left(\frac{-B_0\tau_e}{m_e^*}\right) + \frac{pe^3\tau_h}{m_h^*}\left(\frac{B_0\tau_h}{m_h^*}\right)\right] E_x$$

where $\mu_e = \frac{e\tau_e}{m_e^*}$, $\mu_h = \frac{e\tau_h}{m_h^*}$ as in kinetic theory. Hence, we can write

$$\left(n\mu_e + p\mu_h\right) E_y = B_0 E_x \left(-n\mu_e^2 + p\mu_h^2\right)$$

or

$$\frac{E_y}{E_x B_0} = \frac{-n\mu_e^2 + p\mu_h^2}{n\mu_e + p\mu_h}$$

and $R_H \equiv \frac{E_y}{J_x B_0} = \frac{E_y}{e(\sigma_e + \sigma_h) E_x B_0}$ where $\sigma_e = ne\mu_e$, $\sigma_h = pe\mu_h$

Finally, $$R_H = \frac{-n\mu_e^2 + p\mu_h^2}{e\left(n\mu_e + p\mu_h\right)^2} = \frac{\mu_h^2\left(p - n\frac{\mu_e^2}{\mu_h^2}\right)}{e\mu_h^2\left(p + n\frac{\mu_e}{\mu_h}\right)^2} = \frac{p - nb^2}{e\left(p + nb\right)}, \quad \text{where } b \equiv \frac{\mu_e}{\mu_h}$$

Important point: This result is valid for any orientation of $\vec{B}$ in x-y plane and independent of the particle charge polarity. But the given form of spheroidal constant energy surface is only precise for the electrons in the conduction band of indirect band-gap semiconductors, such as Si and Ge.