7.6:

Use Eq. 7.36 to implement the WKB method:

$$2\int_{x=x_A}^{x=x_B} k_x(x)\,dx = (2m+1)\pi \tag{7.36}$$

Use symmetry to simplify this equation further

$$4\int_{0}^{x=x_B} k_x(x)\,dx = (2m+1)\pi \tag{7.36}$$

$$k_x(x) = k_0\sqrt{n^2(x) - \overline{n}^2}$$

$$n(x) = \begin{cases} \Delta n\left(1 - \frac{2x}{d}\right) + n_{min} & 0 \le x \le \frac{d}{2} \\ \Delta n\left(1 + \frac{2x}{d}\right) + n_{min} & -\frac{d}{2} \le x \le 0 \\ n_{min} & \text{otherwise} \end{cases}$$

Rearrange this equation to solve for x_B (where $n(x) = \overline{n}$):

$$x_B = -x_A = \frac{d}{2}\left(1 - \frac{\overline{n} - n_{min}}{n_{max} - n_{min}}\right)$$

Use computer to numerically plot right side of equation vs. $\overline{n}$. Values of $\overline{n}$ can then be chosen for each mode (i.e. $m = 1, 2, 3, \cdots$).

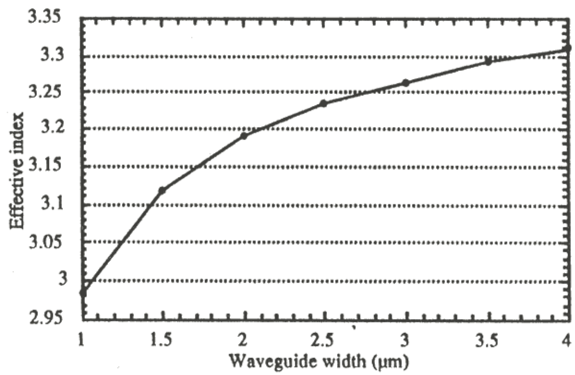

Figure 7.6. Effective index vs. waveguide width

The WKB approximation may not be accurate for $\overline{n}$ near n_{max} or n_{min}.

Plot ray path:

$$\frac{\Delta x}{\Delta z} = \pm \tan\theta_i$$

$$\theta_i = \cos^{-1}\left(\frac{\overline{n}}{n(x)}\right)$$

$d = 2\mu\mathrm{m}$

$\overline{n} = 3.19$ from part (a)

At the center of the waveguide,

$$x = 0$$

$$\theta_i = \cos^{-1}\left(\frac{3.19}{3.5}\right) = 24.2^o$$

So in this case, we can't use the small angle approximation.

At the turnaround point,

$$x = x_B$$

$$n(x) = \overline{n}$$

$$\theta_i = \cos^{-1}\left(\frac{3.19}{3.19}\right) = 0^o$$

$$x_B = 0.124\mu\mathrm{m}$$

To calculate the ray path, step along the path starting at

$$x_0 = 0$$

$$z_0 = 0$$

Using the recursive relation

$$x_{m+1} = x_m + \Delta z \tan\left[\cos^{-1}\left(\frac{\overline{n}}{n(x_m)}\right)\right]$$

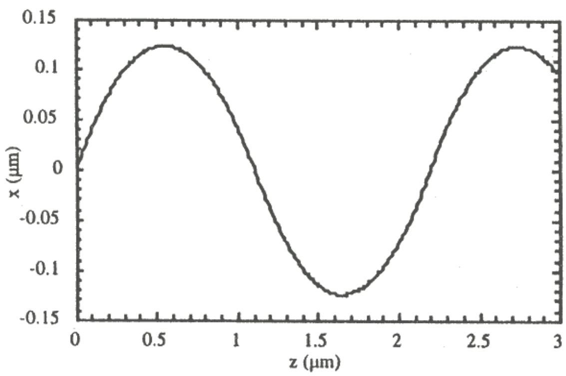

Figure 7.6. Ray path for the case of $d = 2$

Note that the curve is parabolic in shape (i.e. not a sine wave).

7.7:

Use ray tracing to find approximate image distance and magnification. A number of approximations can be made to make the calculation much simpler:

Assume that the index of lens is uniform: $n_{rod} \approx 1.6$

Use small angle approximation, assuming that θ_o and θ_i are small:

$$\rightarrow \qquad \frac{h_o}{d_o} = \tan\theta_o \approx \theta_o$$

$$\rightarrow \qquad \frac{h_i}{d_i} = \tan\theta_i \approx \theta_i$$

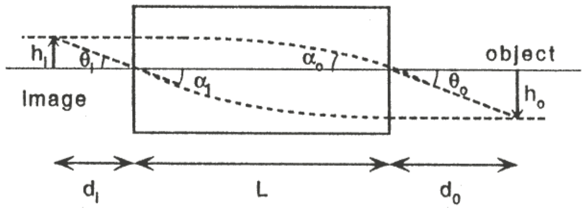

Figure 7.7. Ray tracing for 0.25 pitch GRIN lens

$L = 2$ mm

$d_0 = 1$ mm

By Snell's Law,

$n_{air} \sin\theta_o = n_{rod} \sin\alpha_o$

$n_{air} \sin\theta_i = n_{rod} \sin\alpha_i$

Using small angle approximations and $n_{air} = 1$, these equations reduce to

* $\theta_o = n_{rod}\alpha_o$

* $\theta_i = n_{rod}\alpha_i$

For a 0.25 pitch GRIN lens,

* $h_i = x_o\alpha_o$

* $h_o = x_o\alpha_i$

where $x_0 = L/\left(\frac{\pi}{2}\right)$ as described by Fig. 7.17.

From the approximations above, we saw that

* $\frac{h_o}{d_o} \approx \theta_o$

* $\frac{h_i}{d_i} \approx \theta_i$

Solving the six starred (*) equations for the six unknowns (α_i, α_0, θ_i, θ_0, h_i, and d_i), we find that

$$d_i = \frac{x_o^2}{d_o n_{rod}^2} = \frac{(1.27 \text{ mm})^2}{(1 \text{ mm})(1.6)^2} = 0.63 \text{ mm}$$

$$\text{total distance } = D = d_o + d_i + L = (1 \text{ mm}) + (0.63 \text{ mm}) + (2 \text{ mm}) = 3.63 \text{ mm}$$

$$\text{magnification } = m = -\frac{h_i}{h_o} = -\frac{x_o}{n_{rod}d_o} = -\frac{1.27 \text{ mm}}{(1.6)(1 \text{ mm})} = -0.794$$

7.9:

$$\alpha = \frac{(1-r^2)(m+1)\lambda}{2n_1 d^2} \tag{7.71}$$

$$r = \frac{k_{1x}-k_{2x}}{k_{1x}+k_{2x}} \tag{7.4}$$

For the fundamental mode $m=0$ of an antiguide.

$k_{x1} = (m+1)\frac{\pi}{d} = \frac{\pi}{1\mu m} = 3.14\mu m^{-1}$ (7.69)

$\beta = \sqrt{k_1^2 - k_{x1}^2} = \sqrt{\left(\frac{2\pi(3.2)}{\lambda}\right)^2 - (3.14\mu m^{-1})^2} = 15.244\mu m^{-1}$

$k_{2x} = \sqrt{k_2^2 - \beta^2} = \sqrt{\left(\frac{2\pi(3.4)}{\lambda}\right)^2 - (15.244\mu m^{-1})^2} = 6.38\mu m^{-1}$

$$r = \frac{k_{1x}-k_{2x}}{k_{1x}+k_{2x}} = \frac{3.14-6.38}{3.14+6.38} = 0.34$$

$$\alpha = \frac{(1-0.34^2)(0+1)(1.3\mu m)}{2(3.2)(1\mu m)^2} = 0.180\mu m^{-1} = 1.8\times 10^3\ cm^{-1}$$

7.10:

$$Loss = (Number\ of\ curves)(Length\ of\ curves)(Bend\ loss\ per\ length) = 4L\alpha$$

Find the length of the curves:

From the Fig. 7.29 (in problem statement),

$$R_b - 35\mu\text{m} = R_b \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{R_b - 35\mu\text{m}}{R_b}\right)$$

and

$$L = 2\pi R_b \frac{\theta}{360^o}$$

For $R_b = 100\mu\text{m}$, $\theta = 49.4^o$ and $L = 86.2\mu\text{m}$.

For $R_b = 150\mu\text{m}$, $\theta = 39.9^o$ and $L = 104.4\mu\text{m}$.

Find the bend loss:

$$\alpha = C_1\ e^{-C_2 R_b} \qquad (p.337)$$

Use the effective index method to convert the two dimensional waveguide to a slab waveguide. First deal with the transverse dimension using the method described in Appendix 3:

$V = 1.99$

$a = 0$

$b = 0.46$ estimated from Fig. A3.2.

$\overline{n} = 3.28$

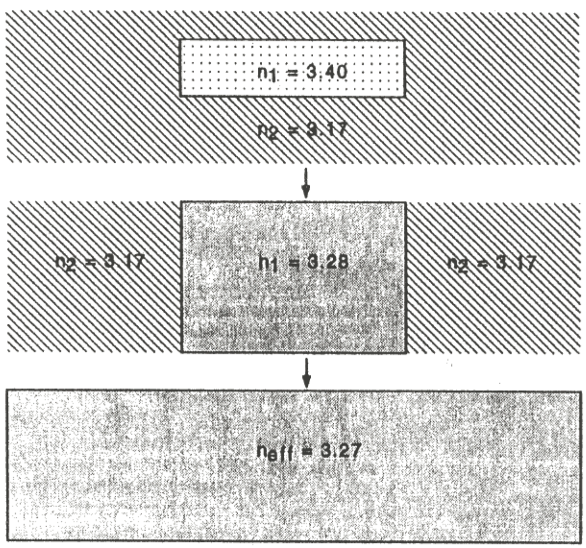

Figure 7.10. Effective index method

Then use effective index method in lateral dimension to find γ_x.

$V = 6.83$

$a = 0$

$b = 0.87$

$\overline{n} = 3.27$

Now find γ_x:

$$\gamma_x = k_0\sqrt{\overline{n}^2 - n_2^2} = \frac{2\pi}{1.55\mu m}\sqrt{3.27^2 - 3.17^2} = 3.25\mu m^{-1}$$

$$k_2 = \frac{2\pi n_2}{\lambda} = \frac{2\pi 3.17}{1.55\mu m} = 12.85\mu m^{-1}$$

$$C_1 = \frac{\gamma_x^2}{k_2} = 0.822\mu m^{-1}$$

$$C_2 = \frac{2}{3}\frac{\gamma_x^3}{k_2^2} = 0.138\mu m^{-1}$$

<u>Calculate the loss</u>:

For $R_b = 100\mu m$,

$$L = 86.2\mu m$$

$$\alpha = (0.822\mu m^{-1})\ e^{-(0.138\mu m^{-1})(100\mu m)} = 8.35\times 10^{-7}\mu m^{-1}$$

$$Loss = 4(8.35\times 10^{-7}\mu m^{-1})(86.2\mu m) = 0.000288 = 0.03\%$$

For $R_b = 150\mu m$,

$$L = 104.4\mu m$$

$$\alpha = (0.822\mu m^{-1})\ e^{-(0.138\mu m^{-1})(150\mu m)} = 8.41\times 10^{-10}\mu m^{-1}$$

$$Loss = 4(8.41\times 10^{-10}\mu m^{-1})(104.4\mu m) = 3.5\times 10^{-7} = 3.5\times 10^{-5}\%$$

7.11:

The method used in Appendix 16 can be followed step by step to solve this problem with Mathematica (see below). Note that the eigenvalue problem expressed by Eq. A16.6 has many eigenvalues (and eigenvectors). The fundamental mode can be found by choosing the largest positive eigenvalue. The field profile $U(x, y)$ will be given by the eigenvector associated with this eigenvalue.

```
dx = 0.1;
dy = 0.4;
lambda = 1.55; (* unit of length is micron *)
k0=2 Pi/lambda;
dX = N[dx k0];
dY = N[dy k0];
jmax = Floor[4.8/dy+1];
imax = Floor[2.3/dx+1];
(***********************)
(* Define Index Matrix *)
(***********************)
n = Table[Table[0,{jmax}],{imax}];
For[y=0;j=1,y<=4.8,y=y+dy;j++,
  For[x=0;i=1,x<=0.65,x = x+dx;i++,
    n[[i,j]]=3.34;
    ];
  For[,x<=0.85,x = x+dx;i++,
    n[[i,j]]=3.44;
    ];
  For[,x<=1.95,x = x+dx;i++,
    If[1.4 < y <= 3.4,
      (*then*) n[[i,j]]=3.44,
      (*else*) n[[i,j]]=1.0
      ];
    ];
  For[,x<=2.3,x = x+dx;i++,
    n[[i,j]]=1.0;
    ];
  ];
(***********************)
(*  Define Variables   *)
(***********************)
(* define a_j^i *)
functiona[i_,j_] := n[[i,j]]^2 - 2/dX^2 - 2/dY^2;
a=Array[functiona,imax,jmax];
(* define b *)
b=1/dY^2;
(* define B *)
B=1/dX^2;
(***********************)
(*   Define Matrix A   *)
(***********************)
matA = IdentityMatrix[imax*jmax];
For[i=1,i<=imax,i++,
  For[j=1,j<=jmax,j++,
    matA[[(i-1)*jmax+j,     (i-1)*jmax+j      ]] = a[[i,j]]
    ];
  For[j=1,j<jmax,j++,
    matA[[(i-1)*jmax+j+1,   (i-1)*jmax+j      ]] = b;
    matA[[(i-1)*jmax+j,     (i-1)*jmax+j+1    ]] = b
    ];
  ];
For[i=1,i<imax,i++,
  For[j=1,j<=jmax,j++,
    matA[[(i-1)*jmax+j+jmax,(i-1)*jmax+j      ]] = B;
    matA[[(i-1)*jmax+j,     (i-1)*jmax+j+jmax]] = B
    ];
  ];
```

Once the matrices have been defined, all of the eigenvalues can be found easily:

```
ans = Eigenvalues[matA]
{-24.4988, -24.4988, -24.1238, -24.1238, -23.8421, -23.8421, -23.6552, -23.6551, -23.4741, -23.4739, -23.2695,
-23.2694, -23.0162, -23.0132, -22.7978, -22.7794, -22.7352, -22.6301, -22.6093, -22.5022, -22.4154, -22.3376,
-22.0981, -21.9388, -21.8675, -21.6847, -21.5377, -21.513, -21.4278, -21.3154, -21.3129, -20.936, -20.9357,
-20.4692, -20.4691, -20.0894, -20.0894, -19.5604, -19.5604, -19.1861, -19.186, -18.7175, -18.7175, -18.3313,
-18.3313, -17.5651, -17.5588, -17.468, -17.2684, -17.1897, -17.1038, -16.8427, -16.7108, -16.6416, -16.4188,
-16.3149, -16.2895, -16.1413, -15.3806, -15.3806, -15.0024, -15.0024, -14.535, -14.5348, -14.165, -14.1535,
-14.1381, -14.0081, -13.9252, -13.8323, -13.6711, -13.5732, -13.5294, -13.372, -13.2995, -13.1707, -13.1072,
-13.1045, -13.0522, -12.9155, -12.9084, -12.8934, -12.788, -12.759, -12.7265, -12.6661, -12.6299, -12.6004,
-12.5111, -12.3196, -12.2589, -12.254, -12.1833, -12.1034, -11.9689, -11.9019, -11.8861, -11.8673, -11.8226,
-11.8095, -11.6231, -11.559, 11.5297, -11.5016, -11.3689, 11.2624, -11.1531, -11.1371, -11.0212, 10.9811, -10.9498,
10.8919, -10.8648, -10.7656, 10.758, -10.6791, -10.6601, -10.6513, -10.6461, 10.5908, 10.5604, 10.4915, -10.49,
10.3901, -10.305, 10.2873, -10.2588, 10.237, 10.2151, -10.2082, 10.1131, -10.0863, 9.99329, 9.9344, -9.93283,
-9.85808, -9.79841, 9.78065, -9.77422, 9.64918, 9.64729, -9.56637, 9.52132, 9.4889, -9.43048, -9.43034, 9.35246,
9.29375, 9.26185, -9.21357, 9.17657, -9.11469, 9.10506, -9.07504, -9.0743, 9.05266, 8.94848, -8.90259, 8.89723,
-8.83531, 8.72152, -8.712, -8.68452, 8.63025, -8.44671, 8.44533, 8.35276, -8.25815, 8.23658, -8.22765, 8.21108,
8.11019, -8.09445, -8.05408, 8.02538, -8.01122, -7.87305, -7.86232, 7.85277, 7.82184, -7.78234, -7.71838, -7.66906,
-7.65262, 7.63883, -7.4264, 7.29001, -7.21238, -7.20193, 7.08208, -7.04245, 7.01975, 6.84667, -6.82881, -6.82473,
-6.67668, 6.61269, 6.55818, 6.45101, -6.41055, -6.31511, -6.3127, 6.2359, -6.14995, 6.1443, 6.00541, -5.95158,
-5.93839, -5.86248, 5.79377, 5.6937, 5.68789, 5.60073, -5.52887, -5.47132, -5.44864, 5.38578, 5.33344, 5.27503,
-5.19514, -5.15276, -5.10375, -5.08788, -5.07358, 4.95147, -4.85774, -4.76566, -4.75093, 4.58766, -4.54749,
-4.34369, 4.31562, -4.30878, -4.23602, -4.04311, 4.0314, -4.02438, -3.94444, -3.92679, -3.86473, 3.78319, -3.67285,
-3.63119, -3.4791, 3.47057, 3.36469, -3.33663, 3.31587, -3.24505, -3.17049, -3.15421, 3.08806, -3.06368, 2.96685,
2.89108, -2.79789, -2.79535, -2.73025, 2.70989, -2.63621, -2.6004, 2.50692, 2.5041, -2.35834, 2.35377, -2.27454,
-2.24418, -2.19137, 2.16139, 2.12976, 2.03676, -2.01751, -2.00978, -1.7704, -1.76911, 1.69062, -1.68933, -1.59863,
-1.55342, -1.40046, -1.40012, 1.32529, -1.31532, -1.2027, -1.1365, -1.08984, -1.08776, 1.05235, -0.955905,
-0.747433, -0.735746, -0.723903, -0.723297, 0.675483, 0.633131, 0.632933, -0.624621, -0.596308, -0.595971,
0.427503, -0.396743, -0.362682, -0.35195, -0.263229, 0.250729, 0.250352, -0.218444, -0.218352, -0.20017, 0.176097,
0.140708, 0.0396406}
```

The eigenvalue corresponding to the fundamental mode is the largest positive eigenvalue: 11.53. This corresponds to an index for the fundamental mode of

$$\overline{n} = 3.396$$

```
value = ans[[103]];
InputForm[value]
nbar = Sqrt[value];
  (* the modal index of the fundamental mode
  is the sqrt of the largest positive eigenvalue *)
InputForm[nbar]
11.52965762343541035
3.395534953941044375
```

Next we solve for the eigenvectors of the system and pick out the eigenvector associated with the eigenvalue chosen for the fundamental mode. This eigenvector gives us the field profile of the fundamental mode.

```
ans = Eigensystem[matA];

vector = ans[[2,103]];

modefield = Table[Table[0,{jmax}],{imax}];
For[i=1,i<=imax,i++,
  For[j=1,j<=jmax,j++,
    modefield[[i,j]] = vector[[(i-1)*jmax+j]];
    ];
  ];

ListContourPlot[modefield, AspectRatio -> 2.3/4.8];
```

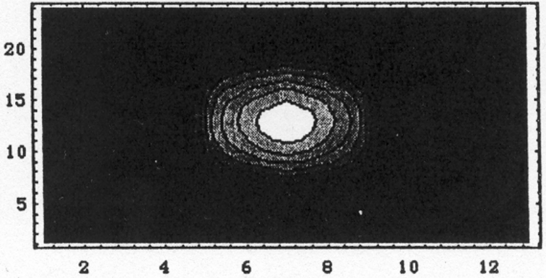

Figure 7.11. A contour plot (to scale) of the field profile for the fundamental mode.

$$\overline{n} = 3.396$$