



Lecture 3 - Wave Propagation in Dielectrics and Basic Dielectric Properties

Maxwell's Equations



$$\nabla \times \bar{h} = i + \frac{\partial d}{\partial t}$$

$$\nabla \times \bar{e} = -\frac{\partial b}{\partial t}$$

$$\nabla \cdot \bar{d} = 0$$

$$\nabla \cdot \bar{b} = 0$$

where $\bar{e}$ and $\bar{h}$ are the electric and magnetic field vectors
 $\bar{d}$ and $\bar{b}$ are the electric and magnetic displacement vectors
No free charge.

Constitutive Relations



$$\vec{d} = \epsilon_0 \vec{e} + \vec{p}$$

$$\vec{b} = \mu_0 (\vec{h} + \vec{m})$$

$\vec{p}$ and $\vec{m}$ are the electric and magnetic polarizations of the medium
 ϵ_0 and μ_0 are the electric and magnetic permeabilities of vacuum
 $\vec{e}$ and $\vec{h}$ are the electric and magnetic field vectors
 $\vec{d}$ and $\vec{b}$ are the electric and magnetic displacement vectors

Electric Susceptibility χ (Isotropic)



Isotropic Media: χ is a complex number

$$\bar{P} = \epsilon_0 \chi_{ij} \bar{E}$$

The real part determines the index (velocity) and the imaginary part determines the gain or absorption.

Isotropic media: Vacuum, gasses, glasses (optical fibers)
Anisotropic media: Semiconductors, crystalline materials.

Electric Susceptibility χ (Anisotropic media)

Anisotropic Media: χ is a complex second rank tensor

$$\bar{P} = \epsilon_0 \chi_{ij} \bar{E}$$

$$\bar{P}_i = \epsilon_0 \sum \chi_{ij} E_j$$

$$P_x = \epsilon_0 (\chi_{xx} E_x + \chi_{xy} E_y + \chi_{xz} E_z)$$

One can always choose a coordinate system such that off axis elements are zero. These are the principal dielectric axes of the crystal. We will only use the principal coordinate system.

$$P_x = \epsilon_0 \chi_{11} E_x$$

$$P_y = \epsilon_0 \chi_{22} E_y$$

$$P_z = \epsilon_0 \chi_{33} E_z$$

Principal Axes

D, E and P are not parallel in general. D and E are related by the electric permeability tensor ϵ

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P}$$

$$\bar{D} = \epsilon \bar{E}$$

Principal axes can always be chosen such that D and E are parallel and the off diagonal elements of ϵ are zero.

$$\epsilon_{11} = \epsilon_0 (1 + \chi_{11})$$

$$\epsilon_{22} = \epsilon_0 (1 + \chi_{22})$$

$$\epsilon_{33} = \epsilon_0 (1 + \chi_{33})$$

Wave Propagation in Lossless, Isotropic Media

- ⇒ Lossless: $\sigma=0$, χ is real, ϵ is real.
- ⇒ Isotropic: χ , ϵ are scalars (not tensors).

$$\nabla \times \bar{e} = i + \frac{\partial \bar{b}}{\partial t} = 0 + \mu \frac{\partial \bar{h}}{\partial t}$$

$$\nabla \times \bar{h} = i + \frac{\partial \bar{d}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{e}) = \mu \frac{\partial (\nabla \times \bar{h})}{\partial t} = \mu \frac{\partial^2 \bar{d}}{\partial^2 t} = \mu \epsilon \frac{\partial^2 \bar{e}}{\partial^2 t}$$

$$\nabla \times (\nabla \times \bar{e}) = \nabla^2 \bar{e} - \nabla (\nabla \cdot \bar{e})$$

$$\nabla^2 \bar{e} = \mu \epsilon \frac{\partial^2 \bar{e}}{\partial^2 t}$$

Wave Equation

Wave Equation



$$\bar{e}(x, y, z, t) = \text{Re}[\bar{E}(x, y, z)e^{i\omega t}]$$

$$\nabla^2 \bar{E} + \omega^2 \mu \epsilon \bar{E} = 0$$

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0$$

where

$$k = \omega \sqrt{\mu \epsilon} = \omega n / c$$

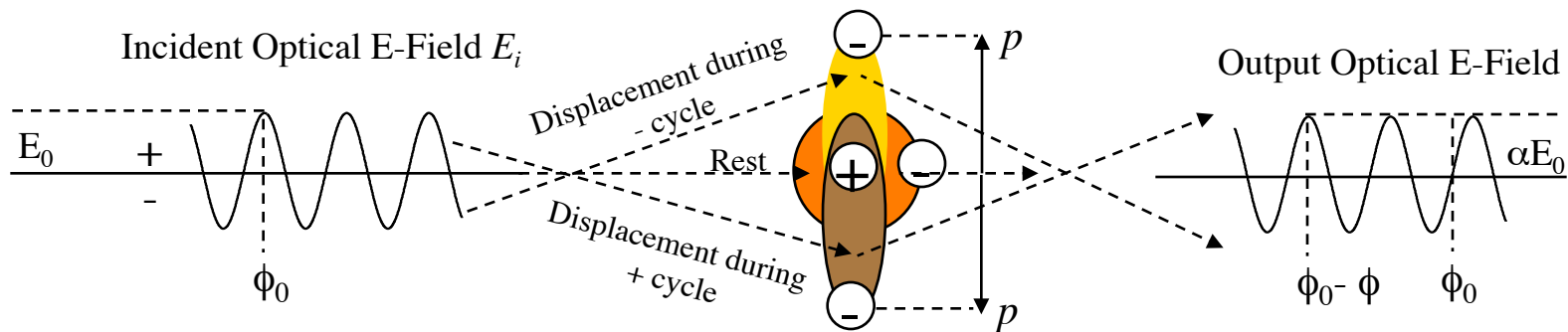
$$c = 1 / \sqrt{\mu_0 \epsilon_0}$$

$$n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

Radiation and Atomic Systems

⇒ Chapter 5, Yariv

⇒ In a dielectric material, an incident optical E-field oscillating at frequency ω will induce physical displacement of a bound electron from its rest state at the same oscillation frequency. We call the the *induced dipole moment or displacement*



α is field attenuation

p is the induced dipole moment

α_p is the atomic polarizability

ϕ_0 is the input field phase reference

ϕ is the output field phase

$$\mathbf{p} = \alpha_p \mathbf{E}_i$$

Dielectric Constant and Index of Refraction

- ⇒ For a homogeneous material of N atoms per unit volume, the polarization is defined by the contribution of all atoms in that volume that interact with the optical E-field

$$\mathbf{P} \approx N\mathbf{p} = N\alpha_p \mathbf{E}_i \equiv \epsilon_0 \chi_e \mathbf{E}_i$$

- ⇒ Where we have defined the *vacuum permittivity* ϵ_0 and *material electric susceptibility* χ_e by

$$N\alpha_p \equiv \epsilon_0 \chi_e$$

- ⇒ Using the constitutive relation $\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$, the *dielectric constant* is defined as

$$\begin{aligned} \epsilon &= \epsilon_0 + \frac{\mathbf{P}}{\mathbf{E}} = \epsilon_0 + \epsilon_0 \chi_e = \epsilon_0 (1 + \chi_e) \\ &= \epsilon_0 \left(1 + \frac{N\alpha_p}{\epsilon_0} \right) \end{aligned}$$

- ⇒ And we can now define for a non-magnetic medium (which dielectrics are at optical frequencies), the *optical index of refraction* n

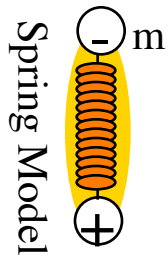
$$\begin{aligned} n^2 &= 1 + \chi_e = 1 + \frac{N\alpha_p}{\epsilon_0} \\ n &= \sqrt{1 + \chi_e} = \sqrt{1 + \frac{N\alpha_p}{\epsilon_0}} \end{aligned}$$

Classical Electron Model

- ⇒ Now we need a basic model that describes the motion, or displacement of the electron (dipole) in the presence of an external force (incident optical field)
- ⇒ Consider the classical equation of motion that describes displacement (position X) of an electron with charge q and mass m attached to an atom (as if connected by a spring) with a dampening coefficient (loss) γ and resonant frequency ω_0 in response to an applied field

$$m \frac{d^2}{dt^2} X + m\gamma \frac{d}{dt} X + m\omega_0^2 X = -qE_a$$

- ⇒ Lets assume the atom can be described as a two level system with energy levels E_1 and E_2 such that the atomic resonance is $\omega_0 = (E_2 - E_1)/\hbar$ and the applied electric field is of the form $E = E_0 e^{i\omega t}$. Then the electron position (relative to rest) and induced dipole moment of the atom can be described as



$$X = \frac{-qE_0}{m(\omega_0^2 - \omega^2 + i\gamma\omega)} e^{i\omega t}$$

$$p = -qX = \frac{-q^2 E_0}{m(\omega_0^2 - \omega^2 + i\gamma\omega)} e^{i\omega t}$$

Classical Electron Model

⇒ The dielectric constant can now be written in terms of the spring model

$$p = \alpha_p E_a$$

$$\alpha_p = \frac{q^2}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

$$n^2 = 1 + \chi = 1 + \frac{N\alpha_p}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

⇒ And we can now re-write the electric susceptibility in terms of the classical electron model and define the real and imaginary parts of χ as χ' and χ''

$$\chi = \chi' - i\chi'' = \frac{Nq^2}{m\epsilon_0(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

Near Resonance Condition



⇒ Near resonance, $\omega \approx \omega_0$, the second term in χ is small compared to the first term and the index of refraction can be approximated by $n=1+ \chi/2$

$$\begin{aligned} n &\approx 1 + \frac{\chi}{2} = 1 + \frac{Nq^2}{2m\epsilon_0(\omega_0^2 - \omega^2 + i\gamma\omega)} \\ &= 1 + \frac{Nq^2}{4\omega_0 m\epsilon_0(\omega_0 - \omega + i\gamma/2)} \end{aligned}$$

Complex Refractive Index and Dispersion

- ⇒ The real and imaginary parts of the electric susceptibility and index of refraction give rise to *optical phase delay* and *optical absorption*.
- ⇒ We can separate the real and imaginary parts by multiplying numerator and denominator by the complex conjugate

$$\begin{aligned}\chi' - i\chi'' &= \frac{Nq^2}{m\varepsilon_0(\omega_0^2 - \omega^2 + i\gamma\omega)} \frac{\omega_0^2 - \omega^2 - i\gamma\omega}{\omega_0^2 - \omega^2 - i\gamma\omega} \\ &= \frac{Nq^2(\omega_0 - \omega)}{2m\omega_0\varepsilon_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]} - i \frac{Nq^2(\gamma/2)}{2m\omega_0\varepsilon_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]} \\ \chi' &= \frac{Nq^2(\omega_0 - \omega)}{2m\omega_0\varepsilon_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]} \\ \chi'' &= \frac{Nq^2(\gamma/2)}{2m\omega_0\varepsilon_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]}\end{aligned}$$

Complex Refractive Index and Dispersion

⇒ The complex index is (not assuming near resonance)

$$n' = n - i\kappa = 1 + \frac{Nq^2(\omega_0^2 - \omega^2)}{2m\epsilon_0[(\omega_0^2 - \omega^2)^2 + (\gamma^2\omega^2)]} - i \frac{Nq^2\gamma\omega}{2m\epsilon_0[(\omega_0^2 - \omega^2)^2 + (\gamma^2\omega^2)]}$$

⇒ For a propagating field, we can write

$$\begin{aligned} E &= Ae^{i(\omega t - k'z)} = Ae^{i(\omega t - \frac{2\pi}{\lambda}(n - i\kappa)z)} \\ &= Ae^{i(\omega t - \frac{2\pi}{\lambda}nz)} e^{-\frac{2\pi}{\lambda}\kappa z} \end{aligned}$$

⇒ Where we can now define the attenuation coefficient α

$$\begin{aligned} \alpha &= \frac{1}{I} \frac{dI}{dz} \\ I(z) &= I(0)e^{-\alpha z} \\ \alpha &= \frac{4\pi}{\lambda}\kappa \end{aligned}$$

Dispersion and Complex Refractive Index

⇒ Looking more closely at n from previous lecture (harmonic oscillator model),

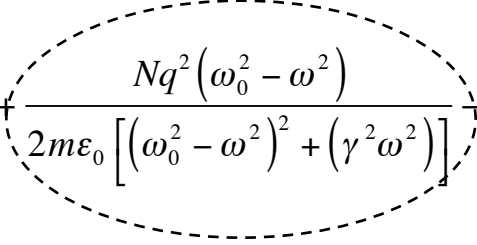
$$n \approx 1 + \frac{\chi}{2} = 1 + \frac{Nq^2}{2m\epsilon_0(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

$$= 1 + \frac{Nq^2}{4\omega_0 m\epsilon_0(\omega_0 - \omega + i\gamma/2)}$$

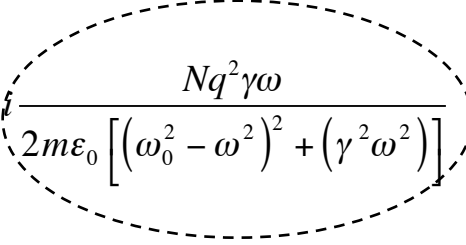
⇒ Using the notation for n as n' to denote a complex number, we see that as ω approaches ω_0 and assuming $\gamma \ll 1$, the index of refraction increases. This change in refractive index as a function of optical frequency is called **Chromatic Dispersion**.

⇒ Looking at the imaginary part of n' we see that the optical E-field will be attenuated as ω approaches ω_0 giving rise to **Optical Absorption**.

$$n' = n - i\kappa = 1 + \frac{Nq^2(\omega_0^2 - \omega^2)}{2m\epsilon_0[(\omega_0^2 - \omega^2)^2 + (\gamma^2\omega^2)]} + i \frac{Nq^2\gamma\omega}{2m\epsilon_0[(\omega_0^2 - \omega^2)^2 + (\gamma^2\omega^2)]}$$



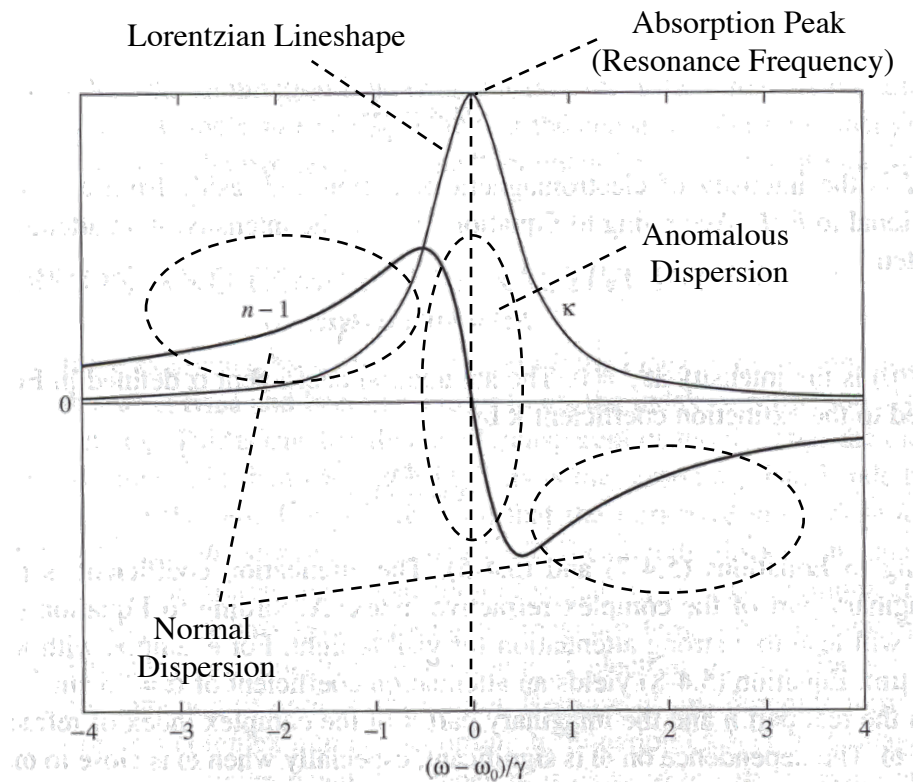
Chromatic Dispersion



Absorption

Dispersion and Complex Refractive Index

- ⇒ Plotting the real ($n-1$) and imaginary (κ) part of the complex refractive index as below, we see the classic relation between index and absorption (gain) for light interaction in atomic systems where atoms are identical.
- ⇒ We note the following characteristics:
 - ⇒ For the imaginary part of the refractive index (κ) the center of the absorption peak is located at ω_0 and falls off symmetrically as ω increase and decreases.
 - ⇒ $\kappa(\omega)$ has a **Lorentzian** lineshape
 - ⇒ The real part of the refractive index (n) approaches unity at low frequencies ($\omega < \omega_0$) and increases and peaks as ω approaches ω_0 . This region where $dn/d\omega > 0$ is called **Normal Dispersion**.
 - ⇒ As ω gets close to ω_0 (very near resonance, where the absorption is high), the index decreases and passes through 1 with increasing frequency. This region where $dn/d\omega < 0$ is called **Anomalous Dispersion**.
 - ⇒ The relationship between the real ($n-1$) and imaginary (κ) part of the complex refractive index is dictated by the **Kramers-Kronig** relationship.



Sellmeir Equation



- ⇒ If we are dealing with a material that has a collection of different types of atoms, each with its own resonance frequency (ω_{oj}), damping factor (γ_j) and fractional per volume (A_j), then we can
 - ⇒ Treat each one as having the absorption and index properties described by the harmonic oscillator model
 - ⇒ Linearly superimpose the absorption and index profiles for each atom for the whole material

$$n^2 = 1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{A_j}{(\omega_j^2 - \omega^2 + i\gamma_j^2\omega)}$$

Loss in early optical fibers

(now the O-H peaks around $1.4 \mu\text{m}$ are small)

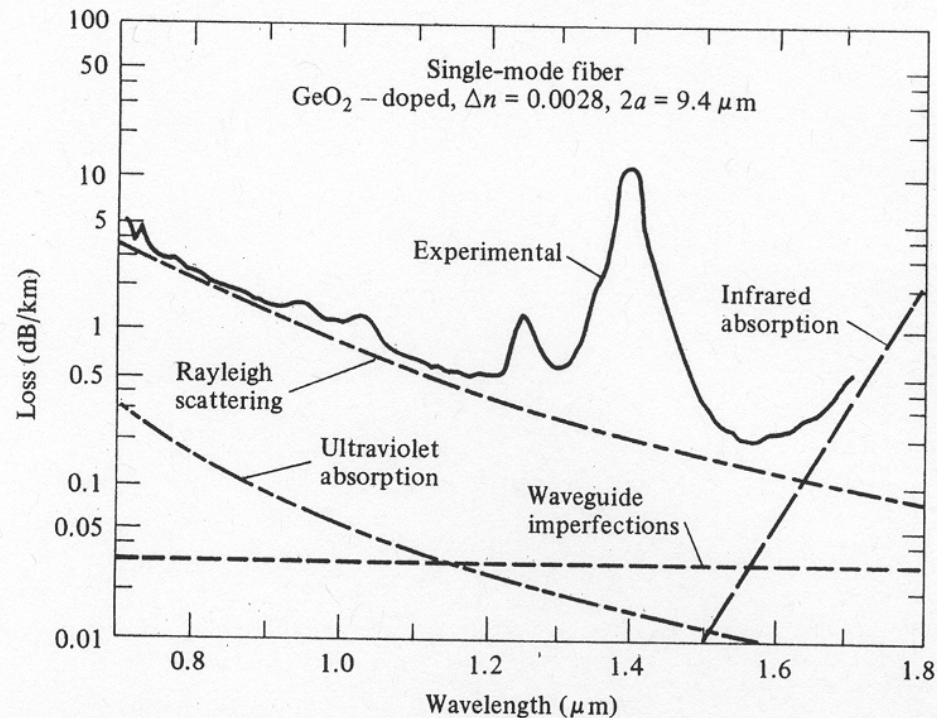


Figure 3-19 Observed loss spectrum of a germanosilicate single-mode fiber. Estimated loss spectra for various intrinsic materials effects and waveguide imperfections are also shown. (From Reference [20].)