## Design Models

small signals → LINEAR

large signals → NONLINEAR

always true?

Important systems NL even for small signals

when true, what is "small"?

NL analysis to VALIDATE linear designs!
Can feedback help?

NL designs improve performance, stability properties, robustness...

new challenges and

MORE FUN!

# Nonlinear vs Linear

## fundamental differences

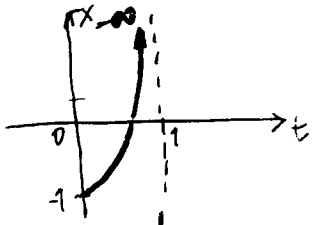
- Nonunique solutions

$$\dot{x} = \sqrt[3]{x}, \quad x(0) = 0$$

i) $x(t) = 0$, ii) $x(t) = \left(\frac{2t}{3}\right)^{3/2}$

- Finite escape time

$$\dot{x} = -x^2, \quad x(0) = -1$$



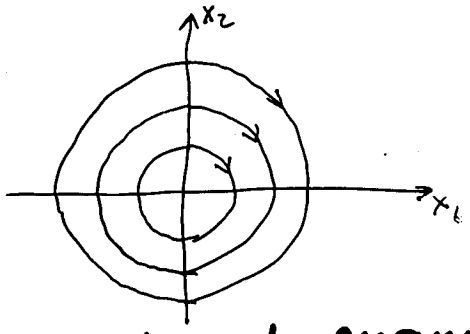

$$-\frac{dx}{x^2} = dt, \quad \frac{1}{x(t)} - \frac{1}{x(0)} = t, \quad x(t) = \frac{1}{t-1}$$

- Multiple isolated equilibria
  many examples to be given
- Limit cycles
  simple periodic orbit ~ impossible in linear systems
- Chaos etc

## Task: Create a limit cycle with feedback control

HARMONIC OSCILLATOR $\ddot{y}+y=0$, $x_1=y$, $x_2=\dot{y}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$
$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$
$$x_2\,dx_2 + x_1\,dx_1 = 0$$
$$r^2 = x_1^2 + x_2^2 = c$$




continuously many!

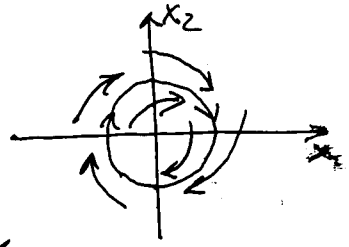
Create only one and converge to it from all initial conditions, say $r=1$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u$$

select

$$u = -x_2(x_1^2 + x_2^2 - 1)$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$




Claim: task completed

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(x_1^2 + x_2^2) = 2x_1x_2 - 2x_2x_1 - 2x_2^2(x_1^2 + x_2^2 - 1)$$

$$\frac{d}{dt}(r^2) \begin{cases} < 0, & r^2 > 1,\ x_2 \neq 0 \\ > 0, & r^2 < 1,\ x_2 \neq 0 \end{cases}$$

Argue that $x_2 = 0$ is not an equilibrium etc

# EXAMPLES OF NONLINEAR MODELS

- Newton, Euler-Lagrange, empirical
- Models in $\mathbb{R}^2$, phase plane plots
- "Sliding modes", Bifurcations
- Model reduction-simplifications
  - Jet engine compressor
  - Hodgkin-Huxley Model
  - Limit cycles

HANDOUTS, JAN 9, 2007

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Keep these examples handy, we will use them on various occasions.

# The Furuta Pendulum

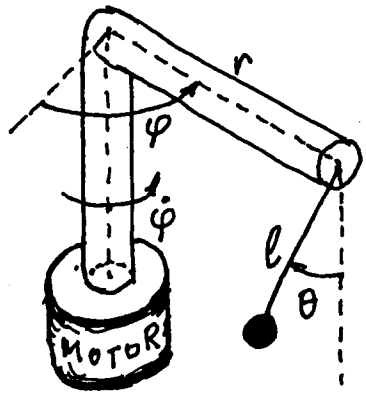
Notes by Prof. K.J.Åström

## Kinetic Energy

$$2T = (J_a + J_p \sin^2\theta)\dot{\varphi}^2 + mr\ell\dot{\varphi}\dot{\theta}\cos\theta + J_p\dot{\theta}^2 \qquad J_p = J + m\ell^2 \qquad J_a = J_b + mr^2$$

## Potential Energy

$$V = mg\ell\cos\theta \qquad L = T - V$$




u = motor torque

$$\frac{\partial L}{\partial \theta} = -mr\ell\dot{\varphi}\dot{\theta}\sin\theta + J_p\dot{\varphi}^2\sin\theta\cos\theta + mg\ell\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = J_p\dot{\theta} + mr\ell\dot{\varphi}\cos\theta$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = mr\ell\dot{\theta}\cos\theta + (J_a + J_p\sin^2\theta)\dot{\varphi}$$

## Equations of motion

$$J_p(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta) + mr\ell\ddot{\varphi}\cos\theta - mg\ell\sin\theta = 0$$

$$mr\ell\ddot{\theta}\cos\theta - mr\ell\dot{\theta}^2\sin\theta + 2J_p\dot{\theta}\dot{\varphi}\sin\theta\cos\theta + (J_a + J_p\sin^2\theta)\ddot{\varphi} = u.$$

$$\omega_0 = \sqrt{\frac{mgl}{J_p}}, \qquad \alpha = \frac{mr\ell}{J_p}, \qquad \beta = \frac{mr\ell}{J_a}, \qquad \frac{1}{\omega_0} \text{ used as time scale, then}$$

the equations become

(FPEq)
$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta + \alpha\ddot{\varphi}\cos\theta - \sin\theta = 0$$

$$\beta\ddot{\theta}\cos\theta - \beta\dot{\theta}^2\sin\theta + 2\frac{\beta}{\alpha}\dot{\theta}\dot{\varphi}\sin\theta\cos\theta + (1 + \frac{\beta}{\alpha}\sin^2\theta)\ddot{\varphi} = u$$

*Automatica*, Vol. 8, pp. 729–736. Pergamon Press, 1972.

# Optimal Control of Bacterial Growth*



G. D'ANS, D. GOTTLIEB and P. KOKOTOVIĆ

**Summary**—A stabilizing feedback control is designed for Haldane–Monod model of microbial growth. The controlled process is optimal in the sense that it approaches the maximum production steady-state from any initial state, while the amount of microorganisms produced in transients is also maximized. Data from a series of experiments are given and used to adjust the model parameters. Satisfactory agreement between experimental and model transients is obtained.

## INTRODUCTION

THE CONTINUOUS cultivation of microorganisms is used in the fermentation industry, in biological degradation of wastes, and in many bacteriological experiments [1–4]. In a continuous cultivation process a fresh nutrient flows into a growth vessel and the culture from the vessel is harvested in a collector. The rate of outflow is equal to the rate of inflow, so that the volume $V$ in the growth vessel remains constant. If the bacterial concentration in the growth vessel is $x$, and the flow rate is $f$, then during an operation period

$$\begin{matrix}\text{the amount}\\ \text{of bacteria}\\ \text{harvested}\end{matrix} = \int_0^T xf\,dt. \tag{1}$$

In many applications an important problem is to find out how to vary the flow rate so as to maximize the amount of bacteria produced.

## ANALYSIS OF THE GROWTH MODEL

The model of microbial growth used in this paper is based on the following assumptions [1, 2, 5, 6].

(1) The bacterial population present in the growth vessel is in exponential phase of growth, that is, each cell is dividing at the same rate,

$$\frac{dx}{dt} = \mu x, \tag{2}$$

where $\mu$ is the specific growth rate, and $x$ the bacteria concentration.

(2) If a substrate metabolized by the microorganism is growth-limiting at lower concentrations and growth-inhibiting at higher concentrations, then the specific growth rate $\mu$ is a function $\mu(s)$ of the substrate concentration $s$, concave for lower, and convex for higher values of $s$. A convenient way to model such a concave–convex function is

$$\mu = \mu(s) = \frac{\mu_m}{1+\frac{K_s}{s}+\frac{s}{K_i}}, \tag{3}$$

where $\mu_m$ is the maximum specific growth rate in absence of inhibition, $K_s$ is the saturation constant [1, 2], and $K_i$ is the inhibition constant [5, 6].

(3) In each cell formed, the same amount of substrate is incorporated. This amount is a constant fraction of the total amount of substrate metabolized,

$$\Delta x = -Y\Delta s, \tag{4}$$

where $\Delta x$ and $\Delta s$ are corresponding variations of $x$ and $s$, and $Y$ is the yield constant.

In a continuous culture, a fresh nutrient in constant concentration $s_r$ flows at a rate $f$ into a growth vessel where the bacteria and the substrate concentrations are $x$ and $s$, respectively. Assuming that no bacteria is present in the nutrient reservoir, the organism balance shows that the change of bacteria concentration equals growth minus outflow,

$$\frac{dx}{dt} = \frac{\mu_m}{1+\frac{K_s}{s}+\frac{s}{K_i}}x - \frac{f}{V}x. \tag{5}$$

The substrate balance shows that the change of substrate concentration equals inflow minus outflow minus substrate metabolized,

$$\frac{ds}{dt} = \frac{f}{V}s_r - \frac{f}{V}s - \frac{1}{Y}\frac{\mu_m}{1+\frac{K_s}{s}+\frac{s}{K_i}}. \tag{6}$$

It is convenient to normalize (5) and (6) by introducing $K_1 = K_s/s_r$ and $K_2 = s_r/K_i$ and the relative units as shown in Table 1.

TABLE 1. RELATIVE UNITS

| Variable | $x$ | $s$ | $f/V$ | $t$ |
|---|---|---|---|---|
| Unit | $Ys_r$ | $s_r$ | $\mu_m$ | $1/\mu_m$ |

Using these units and constants and denoting the normalized flow rate $f/V$ by $u$, (5) and (6) become

$$\frac{dx}{dt} = \left(\frac{s}{K_2s^2+s+K_1}-u\right)x \tag{7}$$

$$\frac{ds}{dt} = u(1-s) - \frac{sx}{K_2s^2+s+K_1}. \tag{8}$$

This is the state-space model of bacterial growth with $x$ and $s$ considered as the state variables and $u$ as the control variable. Another form of this model, with the state variables $z = x+s-1$ and $s$, is

$$\frac{dz}{dt} = -uz \tag{9}$$

$$\frac{ds}{dt} = \left(u - \frac{s}{K_2s^2+s+K_1}\right)(1-s) - \frac{sz}{K_2s^2+s+K_1}. \tag{10}$$

It follows from (9) that the equilibrium states must satisfy $z=0$, that is $x+s=1$. The equilibrium states, that is the roots $(x_o, s_o)$, $(x_1, s_1)$ and $(x_2, s_2)$ of the equations $dx/dt=0$ and $ds/dt=0$ for $u=$ const, are

$$x_o=1-s_o=0, \qquad s_o=1, \tag{11}$$

$$x_1=1-s_1, \qquad s_1=\frac{1-u}{2uK_2}-\tfrac{1}{2}\sqrt{\left(\frac{1-u}{uK_2}\right)^2-4\frac{K_1}{K_2}}, \tag{12}$$

$$x_2=1-s_2, \qquad s_2=\frac{1-u}{2uK_2}+\tfrac{1}{2}\sqrt{\left(\frac{1-u}{uK_2}\right)^2-4\frac{K_1}{K_2}}. \tag{13}$$

Asymptotic stability in the small of these equilibrium states (6) depends on $u$. Consider the realistic case when $K_1<K_2$. Then for $u>(1+2\sqrt{K_1K_2})^{-1}$ the only stable equilibrium is the "wash-out" state (11). For $(1+K_1+K_2)^{-1}<u<(1+2\sqrt{K_1K_2})^{-1}$ both (11) and (12) are asymptotically stable, while (13) is unstable. For $0<u<(1+K_1+K_2)^{-1}$ the only asymptotically stable equilibrium is (12).

In this study, an equilibrium is called the optimum steady-state if it is asymptotically stable in the large and if the amount of bacteria harvested per unit time is maximum. In the steady-state (12) this amount is

$$H=ux=\frac{s(1-s)}{K_2s^2+s+K_1} \tag{14}$$

and it will be maximum, $H=H^*$, if $u=u^*$, where

$$u^*=\frac{s^*}{K_2s^{*2}+s^*+K_1}, \tag{15}$$

is the optimum constant flow and $(x^*, s^*)$,

$$x^*=1-s^*, \qquad s^*=\frac{\sqrt{K_1^2+K_1(1+K_2)}-K_1}{K_2+1} \tag{16}$$

is the corresponding steady-state. To check whether the steady-state $(x^*, s^*)$ is stable in the large the phase-plane portrait of the system (7) and (8) is given in Fig. 1 for $K_1=0{\cdot}1$, $K_2=2{\cdot}0$ and $u=u^*=0{\cdot}51$. The steady-state $(x^*, s^*)$ is asymptotically stable with respect to all the initial states below the separatrix indicated by the dashed line. For the initial states above the separatrix, the wash-out steady-state (11) is asymptotically stable. Thus, if the state $(x, s)$ were above the separatrix, the control $u^*=$const would not return it to $(x^*, s^*)$. Instead, the growth will eventually stop, since the process will approach the wash-out steady-state $x_o=0$, $s_o=1$. To prevent this, the control $u$ must not remain at its constant value $u^*$.

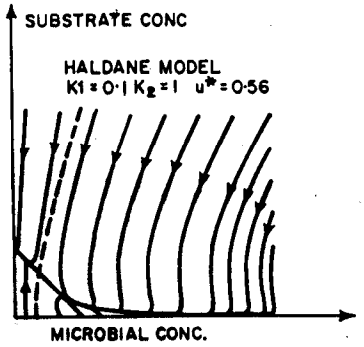



FIG. 1. Phase-plane trajectories of (7), (8) for $K_1=0{\cdot}1$, $K_2=2$ and $u=u^*=0{\cdot}51$.

## OPTIMUM FEEDBACK CONTROL

In this section, a control law is designed to maximize the bacteria production during transient periods and to make the steady-state $(x^*, s^*)$ asymptotically stable in the large. Theoretically, the transients are of infinite duration, that is the interval $(0, T)$ in the performance index (1) is infinite, $T=\infty$. Since the integral (1) gives the total amount of bacteria harvested in both transient and steady-state operations, it is unbounded when $T=\infty$. However, from (14) and (15) the maximum steady-state production is already known to be the integral of $H^*=u^*x^*$ and this amount can be subtracted from (1) to give

$$J=\int_0^\infty (ux-H^*)\mathrm{d}t. \tag{17}$$

This is the incremental amount of bacteria harvested due to a transient, and it can be either positive or negative. A negative value of $J$ corresponds to a loss due to an unfavorable transient, and the maximization of $J$ then means the minimization of the loss.

Our problem is to design a feedback control $u(x, s)$ to maximize $J$ and to force the growth process to the steady-state $(x^*, s^*)$

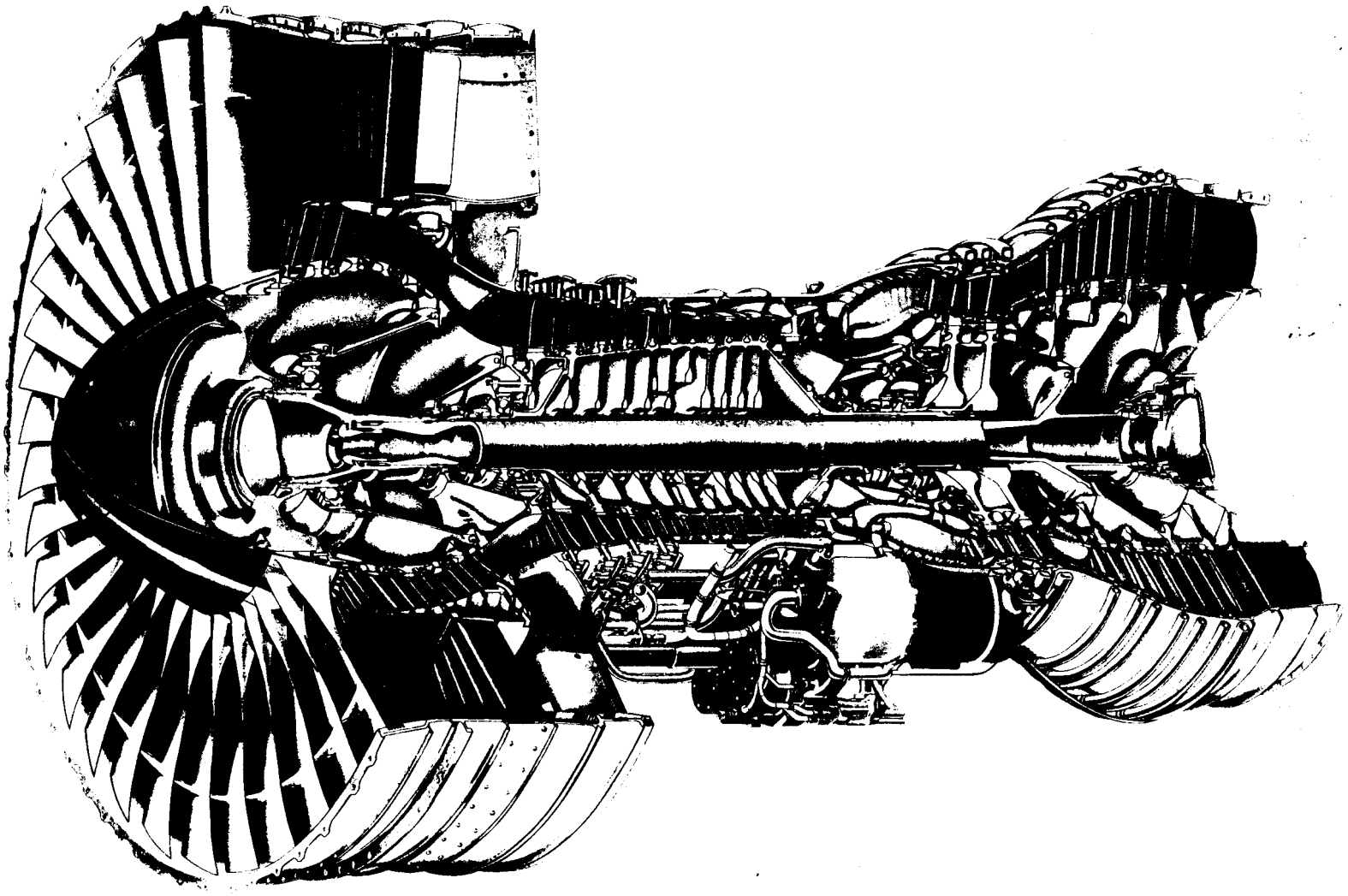
$$\begin{matrix} x(t)\to x^* \\ \qquad\qquad \text{as } t\to\infty \\ s(t)\to s^* \end{matrix} \tag{18}$$

from every initial state $x(0)>0$, $s(0)\geq 0$. Since $u$ represents the flow rate in normalized units, it is physically constrained,

$$0\leq u\leq M=\frac{f}{V} \tag{19}$$

where $f$ is the maximum flow rate permitted by the available equipment.

# PW4000 TURBOFAN



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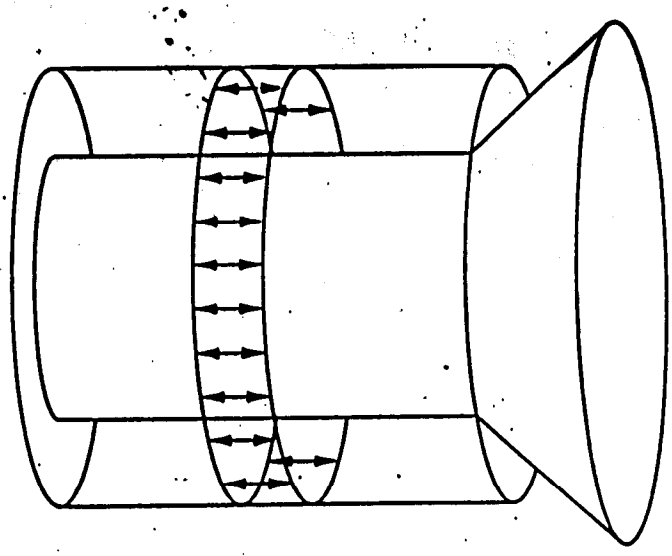
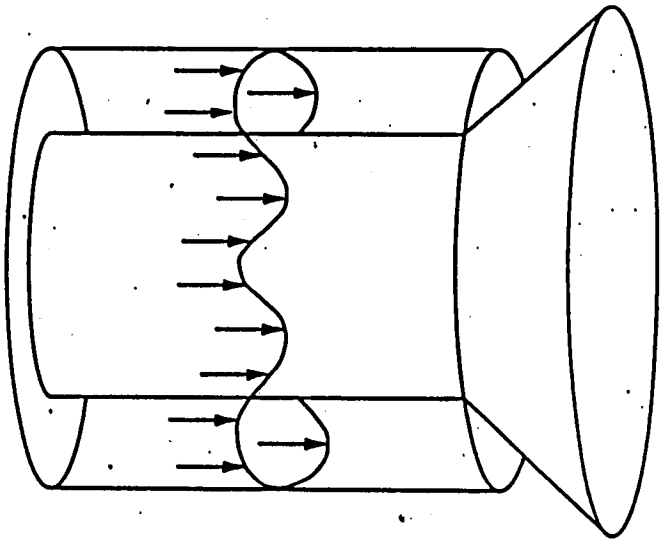
# Compressor Instabilities



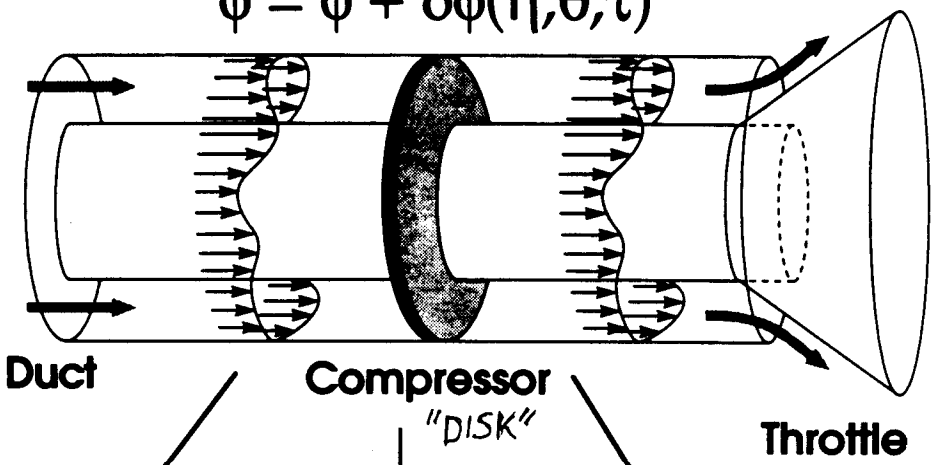
Surge



Rotating Stall

# MOORE-GREITZER: PDEs FOR ROTATING STALL

$$\phi = \bar{\phi} + \delta\phi(\eta,\theta,\tau)$$




**Upstream:**

$$\nabla^2(\delta\Phi) = 0$$

**Downstream:**

$$\nabla^2(\delta P_s) = 0$$

**Across Compressor:**

$$\delta P_s + (m-1)\frac{\partial(\delta\Phi)}{\partial\tau} + \mu\frac{\partial(\delta\phi)}{\partial\tau} + \lambda\frac{\partial(\delta\phi)}{\partial\theta} = \left[\psi_C(\bar{\phi}+\delta\phi)\right]_{n.a.}$$

**Throttle Matching (no surge):**

$$\overline{\psi_C} = \frac{1}{2}K_T\bar{\phi}^2$$

**Moore-Greitzer model:**

$$\begin{aligned}
\dot{\Phi} &= -\Psi + \Psi_C(\Phi) - 3\Phi R \\
\dot{\psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T(\Psi)) \\
\dot{R} &= \sigma R\left(-R - \Phi^2 + 1\right), \quad R(0) \geq 0
\end{aligned}$$

$R$ – normalized stall cell squared amplitude ($R \geq 0$)
$\beta$ – "compliance" coeff. (rotor speed, plenum size)

$$\begin{aligned}
\Psi_C(\Phi) &= \Psi_{C0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3 \\
\Psi &= \frac{1}{\gamma^2}(1 + \Phi_T(\Psi))^2
\end{aligned}$$

$\Phi$ = mass flow
$\Psi$ = pressure rise
$\Phi_T$ = throttle flow
$\gamma$ = throttle opening
$\Psi_{C0}$ = shut-off head coefficient (# of stages)

# Moore-Greitzer PDE Model

$\theta, \eta$=angular, axial coord. $\xi$ norm. time

($\phi'_\eta, \phi'_\xi$ etc. at $\eta = 0$)

$$\Psi(\xi) = \Psi_c(\Phi + \phi'_\eta|_{\eta=0}) - l_c\frac{d\Phi}{d\xi} - m\phi'_\xi|_{\eta=0}$$

$$-\frac{1}{2a}(2\phi'_{\xi\eta} + \phi'_{\theta\eta})|_{\eta=0}$$

$$l_c\frac{d\Phi}{d\xi} = -\Psi(\xi) + \frac{1}{2\pi}\int_0^{2\pi} \Psi_c(\Phi - \phi'_\eta|_{\eta=0})d\theta$$

$$l_c\frac{d\Psi}{d\xi} = \frac{1}{4B^2}(\Phi(\xi) - K_T(\Psi, u))$$

$\phi'$ solves Laplace's equation

$$\phi'_{\eta\eta} + \phi'_{\theta\theta} = 0.$$

# SIMPLE MODEL CAPTURES PHYSICS