ECE 236 PRE-MIDTERM SOLUTIONS 2-20-07

• **4.14**

$$\int_0^{x_1} yg(y)\, dy \geq \int_0^{x_1} y\, dy = \frac{1}{2}x_1^2$$

Therefore,

$$V(x) \geq \tfrac{1}{2}x_1^2 + x_1x_2 + x_2^2 = \tfrac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x$$

The matrix of the quadratic form is positive definite. Hence, $V(x)$ is positive definite for all x, and radially unbounded.

$$\begin{aligned} \dot{V} &= x_1g(x_1)\dot{x}_1 + x_1\dot{x}_2 + x_2\dot{x}_1 + 2x_2\dot{x}_2 \\ &= g(x_1)(x_1x_2 - x_1^2 - x_1x_2 - 2x_1x_2 - 2x_2^2) + x_2^2 \\ &= -g(x_1)(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 \\ &= -g(x_1)x^TQx + x_2^2 \end{aligned}$$

where $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is positive definite. Since $g(x_1) \geq 1$ and $x^TQx \geq 0$, we have

$$\dot{V} \leq -(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 = -(x_1^2 + 2x_1x_2 + x_2^2) = -(x_1 + x_2)^2$$

This shows that $\dot{V}$ is negative semidefinite. We need to apply the invariance principle.

$$\dot{V} = 0 \;\Rightarrow\; 0 \leq -(x_1 + x_2)^2 \;\Rightarrow\; 0 \geq (x_1 + x_2)^2 \;\Rightarrow\; x_1 + x_2 = 0$$

$$x_1(t) + x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_1(t) + \dot{x}_2(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Since $V(x)$ is radially unbounded and all the assumptions hold globally, we conclude that the origin is globally asymptotically stable.

• **4.18** The system has an equilibrium point at $y = Mg/k$ and $\dot{y} = 0$. Let $x_1 = y - Mg/k$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|$$

Take $V(x) = ax_1^2 + bx_2^2$, with $a, b > 0$. $V(x)$ is positive definite and radially unbounded.

$$\dot{V}(x) = 2\left(a - \frac{bk}{M}\right)x_1x_2 - \frac{2bc_1}{M}x_2^2 - \frac{2bc_2}{M}x_2^2|x_2|$$

Taking $a = k/2$ and $b = M/2$, we obtain

$$\dot{V}(x) = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0, \quad \forall\, x$$

Moreover,

$$\dot{V} \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Using LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.

• **4.27** (a) The equilibrium points are the roots of the equations

$$0 = -x_2x_3 + 1, \quad 0 = x_1x_3 - x_2, \quad 0 = x_3^2(1 - x_3)$$

From the third equation, $x_3 = 0$ or $x_3 = 1$. The first equation cannot be satisfied with $x_3 = 0$.

$$x_3 = 1 \;\Rightarrow\; x_2 = 1 \;\Rightarrow\; x_1 = x_2 = 1$$

Hence, there is a unique equilibrium point at $(1,1,1)$.

(b)

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,1,1)} = \left[\begin{array}{ccc} 0 & -x_3 & -x_2 \\ x_3 & -1 & x_1 \\ 0 & 0 & 2x_3 - 3x_3^2 \end{array}\right]_{x=(1,1,1)} = \left[\begin{array}{ccc} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

The eigenvalues are -1 and $(-1 \pm j\sqrt{3})/2$. Hence, the origin is asymptotically stable. The third state equation has equilibrium at $x_3 = 0$. Starting with the initial condition $x_3(0) = 0$, we have $x_3(t) \equiv 0$. Then, $\dot{x}_1 = 1$ and $x_1(t)$ grows unbounded. Thus, the equilibrium point is not globally asymptotically stable.

• 4.28

(a)

$$0 = x_1, \qquad 0 = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2$$

Substitution of $x_1 = 0$ in the second equation yields

$$-x_2(1 + x_2^2) = 0 \Rightarrow x_2 = 0$$

Hence, the origin is the unique equilibrium point.

(b)

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\left[\begin{array}{cc} -1 & 0 \\ (x_2^4 + x_2^2 + 2x_1x_2) & (4x_2^3x_1 - 3x_2^2 + 2x_1x_2 - 1 + x_1^2) \end{array}\right]\right|_{x=0} = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right]$$

Hence, the origin is asymptotically stable.

(c) Let $V(x) = x_1x_2$.

$$\dot{V}(x) = x_1\dot{x}_2 + \dot{x}_1x_2 = (x_1x_2 - 1)x_1x_2^3 + (x_1x_2 - 1 + x_2^2)x_1x_2 - x_1x_2$$

$$\left.\dot{V}(x)\right|_{x_1x_2=2} = 4x_2^2 > 0$$

which implies that Γ is a positively invariant set.

(d) The origin is not globally asymptotically stable since trajectories starting in Γ do not converge to the origin.

• 4.35 If $r_1 \geq r_2$, we have $r_1 + r_2 \leq 2r_1$. Hence,

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) \leq \alpha(2r_1) + \alpha(2r_2)$$

If $r_2 \geq r_1$, we have $r_1 + r_2 \leq 2r_2$. Hence,

$$\alpha(r_1 + r_2) \leq \alpha(2r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

Thus, the inequality

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

is always satisfied.

• 4.36 This exercise is already dealt with in Example 4.18.

• 4.37 (1) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned} \dot{V} &= -x_1^2 + \alpha(t)x_1x_2 + \alpha(t)x_1x_2 - 2x_2^2 \leq -x_1^2 - 2x_2^2 + 2|x_1|\,|x_2| \\ &= -\left[\begin{array}{c} |x_1| \\ |x_2| \end{array}\right]^T \left[\begin{array}{cc} 1 & -1 \\ -1 & 2 \end{array}\right] \left[\begin{array}{c} |x_1| \\ |x_2| \end{array}\right] \leq -0.382(x_1^2 + x_2^2) \end{aligned}$$

where 0.382 is the minimum eigenvalue of the 2×2 matrix. By Theorem 4.10, the origin is exponentially stable.

(2) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V} = -x_1^2 + \alpha(t)x_1x_2 - \alpha(t)x_1x_2 - 2x_2^2 = -x_1^2 - 2x_2^2$$

By Theorem 4.10, the origin is exponentially stable.

• 4.41

(a)

$$x_2 = 1 = \dot{x}_1$$

$$2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 = 2t + 3t + 2 - 3t - 2(t+1) = 0 = \dot{x}_2$$

Thus, $x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is a solution.

(b) Recall from the discussion at the beginning of Section 4.5 that to show asymptotic stability of a solution we shift it to the origin and then show asymptotic stability of the origin. Let $z_1 = x_1 - t$ and $z_2 = x_2 - 1$. Then

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = 2z_1z_2 - z_1 - 2z_2$$

We need to show that the origin $z = 0$ is uniformly asymptotically stable.

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 1 \\ -1 + 2z_2 & -2 + 2z_1 \end{bmatrix}, \quad A = \left.\frac{\partial f}{\partial z}\right|_{z=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

The matrix A is Hurwitz; hence, the origin is uniformly asymptotically stable.

• 4.45 (a) Use $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ as a Lyapunov function candidate.

$$\dot{V} = x_1[h(t)x_2 - g(t)x_1^3] + x_2[-h(t)x_1 - g(t)x_2^3] = -g(t)(x_1^4 + x_2^4) \le -k(x_1^4 + x_2^4)$$

Hence, the origin is uniformly asymptotically stable. Since all assumptions hold globally and $V(x)$ is radially unbounded, the origin is globally uniformly asymptotically stable, which answers part (c).

(b) The conditions of Theorem 4.10 are not satisfied. Let us linearize the system

$$A(t) = \frac{\partial f}{\partial x}(t, 0) = \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$$

Use $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ as a Lyapunov function candidate for the linear system.

$$\dot{V} = x_1h(t)x_2 - x_2h(t)x_1 = 0$$

This shows that solutions starting on the surface $V(x) = c$ remain on that surface for all t. Hence, the origin of the linear system is not exponentially stable. This implies that the origin of the nonlinear system is not exponentially stable.

(d) No.

• 4.54 (1) The system is not input-to-state stable since with $u(t) \equiv c > 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

(3) The system is not input-to-state stable since with $u(t) \equiv 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

• 4.55 (1) Take $V = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned} \dot{V} &= -(x_1^2 + x_2^2) + x_2u \le -\|x\|_2^2 + \|x\|_2|u| \\ &= -(1-\theta)\|x\|_2^2 - \theta\|x\|_2^2 + \|x\|_2|u| \le -(1-\theta)\|x\|_2^2, \ \forall \ \|x\|_2 \ge |u|/\theta \end{aligned}$$

where $0 < \theta < 1$. Hence, the system is input-to-state stable.

(4) With $u = 0$, the system

$$\dot{x}_1 = (x_1 - x_2)(x_1^2 - 1), \quad \dot{x}_2 = (x_1 + x_2)(x_1^2 - 1)$$

has an equilibrium set $\{x_1^2 = 1\}$. Hence, the origin is not globally asymptotically stable. It follows that the system is not input-to-state stable.