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A positive real condition for global stabilization of nonlinear systems

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Abstract: We study the possibility of globally stabilizing, by means of a smooth state feedback, systems obtained by cascading a linear controllable system and a general nonlinear system. Our main result is that global stabilization can be achieved if the output of the linear system can be chosen to be 'feedback positive real' (FPR). Some recent stabilization conditions appear as special cases of the new FPR condition. Examples of systems with the FPR property are given.

Keywords: Stabilization; positive real; feedback positive real; nonlinear control.

1. Introduction

The purpose of this note is to present a sufficient condition for global stabilizability, by means of state feedback, of systems in the form

$$\dot{x} = f(x, \xi), \quad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^\nu, \tag{1.1a}$$

$$\dot{\xi} = A\xi + Bu, \quad u \in \mathbb{R}^m, \tag{1.1b}$$

where:

(H) The pair (A, B) is controllable, the map $f: \mathbb{R}^{n+\nu} \to \mathbb{R}^n$ is smooth (i.e. of class C^∞), and the system

$$\dot{x} = f(x, 0), \quad x \in \mathbb{R}^n,$$

has 0 as a globally asymptotically stable equilibrium.

Our interest in this class of composite systems has been stimulated by normal forms and zero dynamics concepts introduced by Isidori and coworkers [4,2,7]. Such systems may, in general, fail to be globally stabilizable. In [11] an example is exhibited of a system of the above type with the property that there are initial conditions from which *every* trajectory escapes in finite time.

Recent results on global stabilization are of two kinds, namely, those that impose restrictions on the nonlinear system (1.1a), and those where restrictions are imposed on the linear system (1.1b). Results of the first kind were obtained in [8,9,10]. The main result of the second kind, stated and proved in [2,5,13] with slightly different formulations, says that global stabilization of (1.1) is possible when B has rank ν, i.e. in the *relative order one case* [1]. Our theorem is a generalization of the result in [2]. As in [2], the more general stabilization condition presented here gives an explicit formula for the stabilizing feedback, in terms of a Lyapunov function for $\dot{x} = f(x, 0)$. However, unlike the conditions of previous results, ours is not a restriction on one of the two systems (1.1a, b), but a requirement relating them both. We do not impose the condition that the linear system be of relative order one, but require instead that f be of the form

$$f(x, \xi) = f_0(x, \xi) + \sum_i y_i f_i(x, \xi),$$

where f_0 satisfies a stability condition uniformly in ξ and

$$y = C\xi = (y_1, \dots, y_m)^{\mathrm{T}}$$

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[1] In accordance with the terminology introduced by Hirschorn in [3], the 'relative order' of the system (1.1) is the largest controllability index of the linear system (1.1.b).

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– where '$^{\mathrm{T}}$' denotes transpose – is an output that has the 'feedback positive real' (FPR) property, i.e. the property that there be a feedback transformation (in which u is replaced by $K\xi + u$) such that the matrix $A + BK$ is Hurwitz and the transfer function

$$Z_K(s) = C(sI - A - BK)^{-1}B$$

is positive real.

To show how our method applies to a number of cases not covered by other approaches, we study in particular the systems

$$\dot{x} = -(1 + c_1\xi_1 + c_2\xi_2)x^3, \tag{1.2a}$$

$$\dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u, \tag{1.2b}$$

$$\dot{x} = -\left(1 + \alpha\xi_1^2 + 2\beta\xi_1\xi_2 + \gamma\xi_2^2\right)x^3, \tag{1.3a}$$

$$\dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u, \tag{1.3b}$$

where x, ξ_1 and ξ_2 are scalar variables. We determine those values of the parameters c_1, c_2, α, β, γ for which global stabilization by our method is possible. It turns out that, for all other values of these parameters, one can prove that stabilizability fails.

2. The main theorem

First we need some definitions. Consider a differential equation

$$\dot{x} = g(x, \xi), \quad x \in \mathbb{R}^n,$$

that depends on a parameter ξ with values in a set U, and is such that $g: \mathbb{R}^n \times U \to \mathbb{R}^n$ is locally Lipschitz in x for each fixed ξ. We say that a point $p \in \mathbb{R}^n$ is an *asymptotically stable equilibrium, uniformly in* ξ, if p is an equilibrium for each ξ (i.e. $g(p, \xi) = 0$ for each ξ) and there exists a function $V: \mathbb{R}^n \to \mathbb{R}$ that is a strict Lyapunov function for all ξ (that is, V is required to be smooth, proper, strictly positive everywhere except at p, equal to 0 at p, and such that $\nabla V(x) \cdot g(x, \xi) < 0$ for all x such that $x \neq p$).

If $\phi = (\phi_1, \ldots, \phi_k)$ is a k-tuple of smooth real-valued functions on $\mathbb{R}^n$, and F is a real- or vector-valued smooth function on $\mathbb{R}^n$, we say that $f \in \operatorname{span} \phi$ if there are smooth functions $F_1, \ldots, F_k$, with values in the same vector space where F takes values, such that

$$F(z) = \phi_1(z)F_1(z) + \cdots + \phi_k(z)F_k(z) \tag{2.1}$$

for all z.

Recall that an m by m matrix $Z(s)$ of rational functions is said to be *positive real* (PR) if:

(PR1) The entries of $Z(s)$ have no poles s such that $\operatorname{Re} s > 0$.

(PR2) The matrix $Z(s) + Z(s)^*$ (where * denotes conjugate transpose) is nonnegative definite if s is in the imaginary axis and is not a pole of $Z(s)$.

(PR3) The poles of $Z(s)$ that lie on the imaginary axis are all simple, and the corresponding residues are nonnegative definite matrices.

Recall that a transfer matrix

$$Z(s) = C(sI - A)^{-1}B$$

is said to be strictly positive real (SPR) if $Z(s - \varepsilon)$ is PR for some $\varepsilon > 0$. It is well known [7,12] that the above definition is equivalent to the requirement that Z satisfy the following three conditions:

(SPR1) Z has no pure imaginary poles.

(SPR2) $Z(\mathrm{j}\omega) + Z(\mathrm{j}\omega)^* > 0$ for all real ω.

(SPR3) $\lim_{\omega \to \infty} \omega^2 [Z(\mathrm{j}\omega) + Z(\mathrm{j}\omega)^*] > 0$.

Also, let us call Z *weakly strictly positive real* (WSPR) if it satisfies Conditions (SPR1) and (SPR2). A version of the PR Lemma asserts that, if (A, B, C) is minimal and $Z(s) = C(sI - A)^{-1}B$ is PR, then there exist $P > 0$ and $Q \geqslant 0$ such that $A^{\mathrm{T}}P + PA = -Q$ and $PB = C^{\mathrm{T}}$ (cf. [1]). Another version states that, if (A, B) is controllable but (A, C) is not necessarily observable, then the existence of P and Q still follows if $Z(s)$ is SPR (cf. [12]).

We now extend these notions to include the possibility of a state feedback transformation $u \to K\xi + u$. A column

$$y = (y_1, \ldots, y_m)^{\mathrm{T}} = C\xi$$

of linear functions $y_i = c_i\xi$ on $\mathbb{R}^\nu$ (where C is an $m \times n$ matrix, and the row vectors c_i are the rows of C), is a *feedback positive real* (FPR) output of (1.1b) if it has the following property:

(FPR) There exists a linear feedback transformation $u = K\xi + v$ such that $A + BK$ is Hurwitz and there are $P > 0$, $Q \geq 0$ that satisfy

$$(A+BK)^{\mathrm{T}}P + P(A+BK) = -Q$$

and $PB = C^{\mathrm{T}}$.

Then each of the following conditions is sufficient for y to be FPR:

(S1) There is a feedback K such that $A + BK$ is Hurwitz,

$$Z_K(s) = C(sI - A - BK)^{-1}B$$

is PR and $(A + BK, C)$ is observable.

(S2) There is a feedback K such that $A + BK$ is Hurwitz and $Z_K(s)$ is WSPR.

(S3) $m = 1$, $C \neq 0$, and there is a feedback K such that $A + BK$ is Hurwitz and $Z_K(s)$ is PR.

The fact that (S1) suffices follows directly from the Positive Real Lemma. The sufficiency of (S2) would follow directly if instead of 'WSPR' we were requiring 'SPR'. The proof that WSPR suffices is given in [6].

Returning to the system (1.1), we will say that f is *FPR-spanned* if there is an FPR output y of (1.1b) such that $f \in \operatorname{span} y$. A *stable + FPR-spanned* decomposition of f is a decomposition $f = f_0 + \hat{f}$, such that 0 is a globally asymptotically stable equilibrium of $\dot{x} = f_0(x, \xi)$, uniformly in ξ, and $\hat{f}$ is FPR-spanned.

Our main result is:

Theorem 2.1. *If f has a stable + FPR-spanned decomposition, then the system* (1.1) *is globally smoothly stabilizable.*

Proof. First we pick a K for which the matrix $\hat{A} = A + BK$ is Hurwitz, and an output

$$y = (y_1, \dots, y_m)^{\mathrm{T}} = C\xi$$

such that there exist two $n \times n$ real symmetric matrices $P > 0$, $Q \geq 0$ satisfying $\hat{A}^{\mathrm{T}}P + P\hat{A} = -Q$, and $PB = C^{\mathrm{T}}$.

Next we use a decomposition

$$f(x, \xi) = f_0(x, \xi) + y_1 f_1(x, \xi) + \cdots + y_m f_m(x, \xi),$$

where the f_i are smooth and f_0 has 0 as a globally asymptotically stable equilibrium, uniformly in ξ. We then pick a strict Lyapunov function V for $\dot{x} = f_0(x, \xi)$, i.e. a smooth function $V: \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all x such that $x \neq 0$, $V(0) = 0$, $\{x : V(x) \leq \alpha\}$ is compact for every $\alpha \in \mathbb{R}$, and $\nabla V(x) \cdot f_0(x, \xi) < 0$ whenever $x \neq 0$.

We write $u = K\xi + v$, and define

$$W(x, \xi) = V(x) + \xi^{\mathrm{T}}P\xi.$$

We then seek to determine a feedback control $v(x, \xi)$ that will make the origin of $\mathbb{R}^{n+\nu}$ a globally asymptotically stable equilibrium of the closed-loop system

$$\dot{x} = f(x, \xi), \tag{2.2a}$$

$$\dot{\xi} = (A + BK)\xi + v(x, \xi), \tag{2.2b}$$

with Lyapunov function W. The derivative $\dot{W}$ of W along a trajectory of (2.2) is

$$\begin{aligned}\dot{W}(x, \xi) = {} & \nabla V(x) \cdot f_0(x, \xi) \\ & + \sum_{i=1}^{m} y_i \nabla V(x) \cdot f_i(x, \xi) \\ & + \xi^{\mathrm{T}}(\hat{A}^{\mathrm{T}}P + P\hat{A})\xi + 2\xi^{\mathrm{T}}PBv(x, \xi).\end{aligned}$$

Using

$$\hat{A}^{\mathrm{T}}P + P\hat{A} = -Q \text{ and } PB = C^{\mathrm{T}}$$

(so that $\xi^{\mathrm{T}}PB = \xi^{\mathrm{T}}C^{\mathrm{T}} = y^{\mathrm{T}}$), we get

$$\begin{aligned}\dot{W}(x, \xi) = {} & \nabla V(x) \cdot f_0(x, \xi) \\ & + \sum_{i=1}^{m} y_i \nabla V(x) \cdot f_i(x, \xi) \\ & - \xi^{\mathrm{T}}Q\xi + 2\sum_{i=1}^{m} y_i v_i(x, \xi).\end{aligned} \tag{2.3}$$

So, if we choose

$$v_i(x, \xi) = -\tfrac{1}{2}\nabla V(x) \cdot f_i(x, \xi), \tag{2.4}$$

we find that

$$\dot{W}(x, \xi) = \nabla V(x) \cdot f_0(x, \xi) - \xi^{\mathrm{T}}Q\xi.$$

This implies, in particular, that $\dot{W}(x, \xi) \leq 0$ for all x, ξ, and $\dot{W}(x, \xi) < 0$ if $x \neq 0$. Moreover, it is clear that W is smooth and proper, $W(x, \xi) \geq 0$ for all x, ξ, and equality holds if and only if $x = 0$ and $\xi = 0$. To establish the global asymptotic stability of the origin, it suffices to show that, if

$\gamma: t \to (x(t), \xi(t))$ is a complete trajectory of (2.2) along which W is constant, then it follows that $x(t) \equiv 0$ and $\xi(t) \equiv 0$. To begin with, $x(t)$ must be 0 for all t, because $\dot{W}(x, \xi) < 0$ unless $x = 0$. Moreover, once we know that $x(t) \equiv 0$, we can conclude that v vanishes along γ (because $\nabla V(0) = 0$, since V has a minimum at 0). Therefore $t \to \xi(t)$ is a solution of

$$\dot{\xi} = (A + BK)\xi,$$

and

$$W(x(t), \xi(t)) = \xi(t)^{\mathrm{T}} P \xi(t)$$

for all t. Since W is constant along γ, and $\xi(t) \to 0$ as $t \to +\infty$ (because $A + BK$ is Hurwitz), we conclude that $\xi(t)^{\mathrm{T}} P \xi(t) = 0$ for all t. Then, because P is strictly positive definite, it follows that $\xi(t) \equiv 0$. This completes the proof that all the conditions of La Salle's Invariance Principle are satisfied. Therefore the origin is a globally asymptotically stable equilibrium of (2.2). □

3. Examples

We now illustrate the use of Theorem 2.1 by means of a number of examples.

Example 3.1. (*Relative order one.*) A recent stabilization result (cf. [2,5,13]), applies to the special case when B has rank ν. In this case, we can always assume in addition that $m = \nu$. The feedback

$$K = -B^{-1}A - \alpha I$$

(where α is a positive constant), is such that $A + BK = -\alpha I$, and therefore $A + BK$ is Hurwitz. To recover the stabilization result, we take $y = \xi$; then it is clear that the transfer function $Z(s)$ is positive real, because it is a diagonal matrix with entries equal to $1/(s + \alpha)$. Moreover,

$$f(x, \xi) = f(x, 0) + \xi_1 f_1(x, \xi) + \cdots + \xi_\nu f_\nu(x, \xi),$$

for some choice of smooth vector-valued functions f_i. In view of Assumption (H), 0 is a globally asymptotically stable equilibrium of $\dot{x} = f(x, 0)$. By the usual converse Lyapunov theorems, a Lyapunov function $x \to V(x)$ exists. (Uniformity with respect to ξ poses no problem, since $f(x, 0)$ does not depend on ξ.) So all the conditions of Theorem 2.1 are satisfied, and the result of [2,5,13] follows:

Corollary 3.1. *If* (1.1) *satisfies condition* (H) *and B has rank ν, then* (1.1) *is globally smoothly stabilizable.* □

Example 3.2. (*Chain of integrators.*) Suppose a system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m,$$

is stabilizable by a smooth feedback $x \to \bar{u}(x)$. Consider the augmented system

$$\dot{x} = f(x, \xi), \qquad \dot{\xi} = v,$$

obtained by feeding the input v through an integrator. Write $\xi = \bar{u}(x) + \eta$, so that

$$\dot{\eta} = v - D_x\bar{u}(x) \cdot f(x, \bar{u}(x) + \eta).$$

If we let

$$g(x, \eta) = f(x, \bar{u}(x) + \eta),$$

and make the feedback transformation

$$w = v - D_x\bar{u}(x) \cdot f(x, \bar{u}(x) + \eta),$$

we get the system

$$\dot{x} = g(x, \eta), \qquad \dot{\eta} = w,$$

and 0 is a globally asymptotically stable equilibrium of $\dot{x} = g(x, 0)$. So Corollary 3.1 applies, and we come to the conclusion that, if $\dot{x} = f(x, u)$ is smoothly stabilizable, then the augmented system is smoothly stabilizable as well. By induction, it follows that we can augment the original system with any number of integrators, so that we get:

Corollary 3.2. *If a system $\dot{x} = f(x, u)$ is smoothly stabilizable, then any system*

$$\dot{x} = f(x, \xi_1), \quad \dot{\xi}_1 = \xi_2, \ldots, \dot{\xi}_k = v,$$

obtained by cascading the original system with a chain of integrators, is smoothly stabilizable as well. □

Example 3.3. (*Nonminimality.*) To illustrate the use of Condition (S3), consider a system of the form

$$\dot{x} = g(x) + \xi_2 f(x, \xi), \quad x \in \mathbb{R}^n,$$

$$\xi = (\xi_1, \xi_2, \xi_3), \quad \dot{\xi}_1 = \xi_2,$$

$$\dot{\xi}_2 = u, \quad \dot{\xi}_3 = \alpha\xi_1 + \beta\xi_3.$$

We assume that 0 is globally asymptotically stable for $\dot{x} = g(x)$. The three-dimensional linear system is controllable as long as $\alpha \neq 0$. If we take the output to be ξ_2, then the state ξ_3 is unobservable. Nonetheless, the feedback

$$u = -a^2\xi_1 - 2a\xi_2 + v$$

drives ξ_1 and ξ_2 to zero and makes the transfer function $v \to \xi_2$ positive real, provided that $a > 0$. The numerator of the transfer function has two zeros: $s = \beta, 0$. The denominator has a simple zero at $s = \beta$ and a double zero at $s = -a$. So, if $\beta < 0$, Condition (S3) is satisfied, and therefore global stabilization is possible.

Notice that in this case the linear system is unobservable but detectable, and it is possible to proceed directly, by disregarding unobservable states. Then the construction of Theorem 2.1 furnishes a feedback for which $W(x, \xi) = V(x) + \xi^T P \xi$, where $\xi^T P \xi$ is a strictly positive definite function of ξ_1 and ξ_2 only. Although W is not a true Lyapunov function, because it is only positive semidefinite, the fact that W decreases along trajectories of the closed-loop system suffices to prove that the (x, ξ_1, ξ_2) part of every trajectory remains bounded as $t \to \infty$. Since $\beta < 0$, the system $\dot{\xi}_3 = \alpha\xi_1 + \beta\xi_3$ is BIBS if we regard ξ_1 as the input, so ξ_3 will remain bounded as well. One can then prove that all trajectories go to zero exactly as if W were a true Lyapunov function.

If $\beta > 0$, then the linear system is not detectable and the above approach does not work. Any feedback which makes the transfer function $v \to \xi_2$ positive real accomplishes this by cancelling the zero at $s = \beta > 0$ by an unstable pole, that is an unstable eigenvalue of $A + BK$. Hence the output ξ_2 is not FPR.

The last two examples give a complete characterization of the stabilizability region in parameter space, in the sense that, for those values of the parameters where our conditions fail, we are able to prove that stabilizability fails as well, and does so in a very dramatic way, namely, there are initial conditions from which *every* trajectory is unbounded.

Example 3.4. (*Stability vs. unboundedness.*) We prove the following claim:

(A) *The system* (1.2) *is smoothly stabilizable if and only if* $c_1c_2 \geq 0$.

Let us consider first the case when $c_1c_2 \geq 0$ and $c_2 \neq 0$. Then we might as well assume that $c_1 \geq 0$ and $c_2 > 0$ for, if c_1 and c_2 were negative, then we could change ξ_1, ξ_2 and u into $-\xi_1$, $-\xi_2$ and $-u$.

Let $p(s) = c_1 + c_2 s$. If we choose a feedback $u = -k_1\xi_1 - k_2\xi_2 + v$ that makes the polynomial

$$q(s) = s^2 + k_2 s + k_1$$

Hurwitz, and let $y = c_1\xi_1 + c_2\xi_2$ be the output, then observability will follow provided that $q(s)$ and $p(s)$ have no common zeros. In particular, we can choose $k_1 = a^2$ and $k_2 = 2a$, where $a > 0$ is such that $c_1 - ac_2 \neq 0$. It then follows that the transfer function $Z(s) = p(s)/q(s)$ is positive real, provided that $2ac_2 > c_1$. We can then write the right-hand side of the first equation of (1.2) as $f_0 + yf_1$, where $f_0 = f_1 = -x^3$. Clearly, 0 is an asymptotically stable equilibrium for f_0. So all the conditions of our Theorem 2.1 are satisfied. Therefore (1.2) is smoothly stabilizable. Moreover, we can obtain explicit formulas for the stabilizing feedbacks by just carrying out the construction of the proof of Theorem 2.1. The resulting feedback turns out to be given by

$$u(x, \xi) = -a^2\xi_1 - 2a\xi_2 + x^4,$$

where a is required to satisfy $2ac_2 > c_1$.

To complete our discussion of the case when $c_1c_2 \geq 0$, we need to consider the possibility that $c_2 = 0$. However, in this case the system (1.2) is obtained as a cascade of the system $\dot{x} = -(1 + \eta)x^3$, with η being an input, preceded by a chain of two integrators. So Corollary 3.2 applies, and (1.1) is smoothly stabilizable in this case as well.

We now turn to the case $c_1c_2 < 0$, when Theorem 2.1 does not apply. In this case, we may assume that $c_1 < 0$, and then $c_2 > 0$. Write $c_1 = -\rho$, so the first equation of (1.2) becomes

$$\dot{x} = -(1 - \rho\xi_1 + c_2\xi_2)x^3, \tag{3.1}$$

with $c_2 > 0$ and $\rho > 0$. Let

$$t \to (x(t), \xi_1(t), \xi_2(t))$$

be the trajectory of our system corresponding to some open-loop control $t \to u(t)$, defined on an interval $I \subseteq [0, \infty)$ that contains 0, and starting at an initial condition

$$p = (x(0), \xi_1(0), \xi_2(0)).$$

Let $x(0)^2 = 1/2\beta$, $\alpha = \xi_1(0)$. Then $x(t)^2$ can be written explicitly using (3.1), and we get:

$$x(t)^2 = \frac{1}{2\beta + 2\phi(t)}, \tag{3.2}$$

where

$$\phi(t) = t - \rho\int_0^t \xi_1(t)\ \mathrm{d}t + c_2\int_0^t \xi_2(t)\ \mathrm{d}t.$$

Since $\dot{\xi}_1 = \xi_2$, we have

$$\phi(t) = t + c_2[\xi_1(t) - \xi_1(0)] - \rho\int_0^t \xi_1(t)\ \mathrm{d}t,$$

and then

$$2x(t)^2 = \frac{1}{\beta + t - c_2\alpha + c_2\xi_1(t) - \rho\int_0^t \xi_1(s)\ \mathrm{d}s}$$

$$\stackrel{\text{def}}{=} \frac{1}{\psi(t)}. \tag{3.3}$$

Clearly $\psi(t) > 0$ for all $t \in I$. Therefore

$$c_2\xi_1(t) \geq c_2\alpha - \beta - t + \rho\int_0^t \xi_1(s)\ \mathrm{d}s$$

for $t \in I$. This implies that

$$c_2\zeta(t) \geq \theta + \rho\int_0^t \zeta(s)\ \mathrm{d}s,$$

where $\zeta(t) = \xi_1(t) - 1/\rho$ and $\theta = c_2\alpha - \beta - c_2/\rho$. If we let

$$\sigma(t) = \theta + \rho\int_0^t \zeta(s)\ \mathrm{d}s,$$

we see that $\dot{\sigma}(t) \geq (\rho/c_2)\sigma(t)$, so that, if

$$\omega(t) = \sigma(t)\ \mathrm{e}^{-(\rho/c_2)t},$$

then

$$\dot{\omega}(t) = \left(\dot{\sigma}(t) - \frac{\rho}{c_2}\sigma(t)\right)\mathrm{e}^{-(\rho/c_2)t} \geq 0,$$

$$\omega(t) \geq \omega(0) \quad \text{for all } t \in I,$$

which implies that

$$\sigma(t)\ \mathrm{e}^{-(\rho/c_2)t} \geq \sigma(0) = \theta.$$

Thus, for $t \in I$, we have $\sigma(t) \geq \theta\mathrm{e}^{(\rho/c_2)t}$, so that $\zeta(t) \geq (\theta/c_2)\ \mathrm{e}^{(\rho/c_2)t}$ and then, finally

$$\xi_1(t) \geq \frac{1}{\rho} + \frac{\theta}{c_2}\ \mathrm{e}^{(\rho/c_2)t}.$$

If $\theta > 0$, and I is infinite, then the above inequality implies that the function $t \to \xi_1(t)$ is unbounded as $t \to \infty$. So we have proved:

(A′) *If $c_1 < 0$, $c_2 > 0$, and $p \in \mathbb{R}^s$ is an initial condition for* (1.2) *such that $p = (\bar{x}, \alpha, \bar{\alpha})$, $\bar{x} \neq 0$, and*

$$\alpha > \frac{1}{2c_2\bar{x}^2} - \frac{1}{c_1}, \tag{3.4}$$

then every trajectory

$$t \to (x(t), \xi_1(t), \xi_2(t))$$

of (1.2) *that starts at p is such that either $\xi_1(t)$ is unbounded as $t \to \infty$ or $x(t)$ escapes to infinity in finite positive time.*

This clearly implies that, if $c_1c_2 < 0$, then (1.2) *fails to be asymptotically controllable to zero*, so *a fortiori* this system fails to be smoothly stabilizable, and the proof of (A) is complete. □

Example 3.5. (*Stability vs. unboundedness.*) We show the following claim:

(B) *If $\alpha < 0$, $\gamma \leq 0$, and $\beta \geq -\sqrt{\alpha\gamma}$, then the system* (1.3) *has initial conditions p with the property that every trajectory*

$$t \to (x(t), \xi_1(t), \xi_2(t))$$

of (1.3) *that starts at p is such that either $\xi_1(t)$ is unbounded as $t \to \infty$ or $x(t)$ escapes to infinity in finite positive time. For all other values of the parameters α, β, γ,* (1.3) *is smoothly stabilizable.*

We let

$$q(\xi_1, \xi_2) = \alpha\xi_1^2 + 2\beta\xi_1\xi_2 + \gamma\xi_2^2.$$

To prove (B), we first consider the stabilizable cases. Let

$$f(x, \xi) = -(1 + q(\xi_1, \xi_2))x^3.$$

If $\alpha \geq 0$, then we can write

$$f(x, \xi) = f_0(x, \xi) + \xi_2 f_1(x, \xi),$$

where

$$f_0(x, \xi) = -(1 + \alpha\xi_1^2)x^3$$

and

$$f_1(x, \xi) = -(2\beta\xi_1 + \gamma\xi_2)x^3.$$

Using the Lyapunov function $V(x) = x^2$, it is clear that 0 is a globally asymptotically stable equilibrium of $\dot{x} = f_0(x, \xi)$, uniformly in ξ. It is easy to verify that ξ_2 is FPR, so Theorem 2.1 applies.

If $\gamma > 0$, then we can find $a > 0$ such that

$$b = \gamma a^2 - 2\beta a + \alpha > 0.$$

If we let $\eta = \xi_2 + a\xi_1$, then η is FPR. On the other hand,

$$q(\xi) = b\xi_1^2 + \eta(\gamma\eta + 2(\beta - \gamma a)\xi_1),$$

so we can write

$$f(x, \xi) = f_0(x, \xi) + \eta f_1(x, \xi),$$

where

$$f_0(x, \xi) = -(1 + b\xi_2^2)x^3$$

and

$$f_1(x, \xi) = -(\gamma\eta + 2(\beta - \gamma a)\xi_1)x^3.$$

Once again, 0 is a globally asymptotically stable equilibrium of $\dot{x} = f_0(x, \xi)$, uniformly in ξ, and Theorem 2.1 applies.

Next assume that $\alpha < 0$ and $\gamma \le 0$, and write $\alpha = -\delta^2$, $\gamma = -\varepsilon^2$, with $\delta > 0$, $\varepsilon \ge 0$, so that

$$q(\xi) = -\delta\xi_1^2 + 2\beta\xi_1\xi_2 - \varepsilon^2\xi_2^2.$$

Completing the square yields

$$q(\xi) = -\eta^2 + \theta\xi_2^2,$$

where

$$\eta = \delta\xi_1 - (\beta/\delta)\xi_2 \quad \text{and} \quad \theta = (\beta^2/\delta^2) - \varepsilon^2.$$

If $\beta < 0$ and $\theta \ge 0$ (i.e if $\beta < 0$ and $\beta^2 \ge \gamma\alpha$), then η is FPR, and we can write

$$f(x, \xi) = f_0(x, \xi) + \eta f_1(x, \xi),$$

with

$$f_0(x, \xi) = -(1 + \theta\xi_2^2)x^3$$

and

$$f_1(x, \xi) = \eta x^3.$$

So our theorem again applies.

Finally, the case $\alpha < 0$, $\gamma \le 0$, $\beta = 0$, $\theta \ge 0$, can only arise when $\varepsilon = 0$, i.e. $q(\xi) = \alpha\xi_1^2$, since $\beta^2 \ge \delta^2\varepsilon^2$. In this case, smooth stabilizability follows from Corollary 3.2. This completes the consideration of all the cases in which (1.3) has been asserted to be stabilizable.

To analyze the remaining case, we assume that $\alpha = -\delta^2$, $\gamma = -\varepsilon^2$, with $\delta > 0$, $\varepsilon \ge 0$, and either $\theta < 0$ or $\beta > 0$. We will consider separately the two (non-mutually exclusive) subcases: $\theta < 0$ and $\beta > 0$.

If $\theta < 0$, i.e. $\beta^2 < \delta^2\varepsilon^2$ (so that $\varepsilon > 0$), then

$$q(\xi) = -\delta^2\xi_2 - \varepsilon^2\xi_2^2 + 2\beta\xi_1\xi_2.$$

Consider a trajectory

$$t \to (x(t), \xi_1(t), \xi_2(t))$$

of (1.3), defined on an interval $I \subseteq [0, \infty)$ such that $0 \in I$. Let $x(0)^2 = 1/2A$. Then

$$2x(t)^2 = \frac{1}{A + \phi(t)},$$

where

$$\phi(t) = t + \int_0^t q(\xi_1(s), \xi_2(s))\, ds.$$

We will show that there are initial conditions $x(0)$ and $\xi_1(0)$ from which every trajectory escapes to infinity in time not greater than 1. This will follow if one can choose the initial condition so as to ensure that $\phi(1) + A < 0$. But $\phi(1) = 1 - \psi(1)$, where

$$\psi(t) = \int_0^t \left[\delta^2\xi_1(s)^2 + \varepsilon^2\xi_2(s)^2 - 2\beta\xi_1(s)\xi_2(s)\right] ds. \tag{3.5}$$

If $\phi(1) + A \ge 0$, then $\psi(1) \le A + 1$. Fix a number $\rho > 0$ such that $|\beta|/\varepsilon^2 < \rho < \delta^2/|\beta|$. (Such a ρ exists because $\beta^2 < \delta^2\varepsilon^2$.) Then

$$-2\beta\xi_1(s)\xi_2(s) \ge -\rho|\beta|\xi_1(s)^2 - \frac{|\beta|\xi_2(s)^2}{\rho}$$

for all s. So

$$\lambda\int_0^1 \xi_1(s)^2\, ds + \mu\int_0^1 \xi_2(s)^2\, ds \le A + 1, \tag{3.6}$$

where $\lambda = \delta^2 - \rho|\beta|$ and $\mu = \varepsilon^2 - |\beta|/\rho$, so that $\lambda > 0$ and $\mu > 0$. This implies in particular that

$$\int_0^1 \xi_2(s)^2 \, ds \le (A+1)/\mu.$$

Therefore, since $\xi_2 = \dot{\xi}_1$, we have the bound

$$|\xi_1(t) - \xi_1(0)| \le \int_0^1 |\xi_2(s)| \, ds \le \left(\frac{A+1}{\mu}\right)^{1/2}$$

for $0 \le t \le 1$, where the last step follows from Schwartz' inequality. If $|\xi_1(0)| = B$, then the preceding inequality implies in particular that

$$|\xi_1(t)| \ge B - \left(\frac{A+1}{\mu}\right)^{1/2} \tag{3.7}$$

for $0 \le t \le 1$. Now suppose that B is chosen so that

$$B > \left(\frac{A+1}{\mu}\right)^{1/2} + \left(\frac{A+2}{\lambda}\right)^{1/2}. \tag{3.8}$$

Then (3.7) implies that $|\xi_1(t)| \ge ((A+2)/\lambda)^{1/2}$ for $t \in [0, 1]$. If we square both sides, integrate from 0 to 1, and multiply by λ, we get

$$\lambda \int_0^1 \xi_1(s)^2 \, ds \ge A + 2. \tag{3.9}$$

This clearly contradicts (3.6). The contradiction shows that, as long as the initial condition

$$p = (x(0), \xi_1(0), \xi_2(0))$$

is chosen so that (3.8) holds, then $\phi(1) + A$ must be negative for *every* trajectory that starts at this initial condition. So every trajectory that starts at such a p escapes to infinity in time ≤ 1.

To conclude, we consider the remaining nonstabilizable subcase, i.e. when $\beta > 0$ (assuming always that $\alpha < 0$ and $\gamma \le 0$). Our reasoning here will be similar to the one used for the nonstabilization part of the analysis of (1.2). For a suitable choice of initial conditions, and using ϕ, ψ, A as before, we will show that, if the function $t \to \xi_1(t)$ is bounded for $0 \le t < \infty$, then $\phi(t) + A$ must be negative for some $t > 0$, so that the x component of the trajectory escapes to infinity in finite time. If this were not true, then the bound $\psi(t) \le t + A$ would have to hold for all nonnegative t. Therefore

$$\delta^2 \int_0^t \xi_1(s)^2 \, ds - 2\beta \int_0^t \xi_1(s)\xi_2(s) \, ds + \varepsilon^2 \int_0^t \xi_2(s)^2 \, ds \le A + t \tag{3.10}$$

for $t \ge 0$. In particular,

$$2\beta \int_0^t \xi_1(s)\xi_2(s) \, ds \ge \delta^2 \int_0^t \xi_1(s)^2 \, ds - t - A.$$

Since $2\xi_1\xi_2$ is the derivative of ξ_1^2, and $\beta > 0$, we can conclude that

$$\xi_1(t)^2 - \xi_1(0)^2 \ge \frac{\delta^2}{\beta} \int_0^t \xi_1(s)^2 \, ds - \frac{t}{\beta} - \frac{A}{\beta}. \tag{3.11}$$

With $\sigma(t) = \xi_1(t)^2 - 1/\delta^2$, inequality (3.11) can be rewritten as

$$\sigma(t) \ge \frac{\delta^2}{\beta} \int_0^t \sigma(s) \, ds + \sigma(0) - \frac{A}{\beta}. \tag{3.12}$$

If $\sigma(0) > A/\beta$, then (3.12) implies that $\sigma(t) \to \infty$ as $t \to \infty$, and hence $t \to \xi_1(t)$ is unbounded. So, if $t \to \xi_1(t)$ is bounded, this contradicts the assumption that $\phi + A$ is never negative. Thus $\phi + A$ does become negative, and the trajectory escapes in finite time. We have therefore established that every trajectory from any initial condition $(x(0), \xi_1(0), \xi_2(0))$ such that

$$\xi_1(0)^2 > \frac{1}{|\alpha|} + \frac{1}{2\beta x(0)^2}$$

is either unbounded or escapes to infinity in finite time. The proof of (B) is now complete. □

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