# Global Asymptotic Stabilization of the Ball–and–Beam System [1]

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**Abstract**

For the well known ball–and–beam system, we design a new saturation control law, which employs state-dependent saturation levels and guarantees global asymptotic stability, not achieved with previous designs.

## 1 Introduction

The ball–and–beam system poses a challenging stabilization problem, representative of the difficulties introduced by rapidly growing nonlinearities. For the ball, the critical nonlinearity is the centrifugal force $x_1x_4^2$, where $x_1$ is the ball's displacement from its equilibrium and $x_4$ is the beam's angular velocity. This force is destabilizing when it opposes the controlling gravitational force $-\sin x_3$, where $x_3$ is the beam's angle. The simplified equations are:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1x_4^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u, \end{aligned} \tag{1.1}$$

where $(x_1, x_2, x_3, x_4) \in I\!R \times I\!R \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times I\!R$, and the beam's acceleration is considered to be the control variable. This model was introduced in [1], where approximate feedback linearization designs were shown to improve a standard Jacobian linearization design. As pointed out in [2], the controller must not be allowed to cause "peaking" in $x_4$, because then $x_1x_4^2$ may dominate the term $\sin x_3$. Teel [5, 6] prevented peaking with a nested saturation design, which, in the presence of friction, resulted in global asymptotic stability. Without friction, the stability property is semiglobal, whereby with smaller saturation levels larger regions of attraction can be achieved. The destabilizing effect of peaking is thus limited by the low-gain character of the control law, in which the control magnitude never exceeds a small constant.

In this paper we achieve *global* stabilization by letting some of the saturation levels be state-dependent, similar to the low-gain design for nonaffine systems with marginally stable free dynamics [4].

While our control law preserves the structure and simplicity of Teel's control law, the analysis with the state-dependent saturation levels is more complex and requires tighter solution estimates.

## 2 Main Result

Treating $x_4$ in (1.1) as a virtual control $v$, we first globally asymptotically stabilize the abridged system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1v^2 \\ \dot{x}_3 &= v, \end{aligned} \tag{2.1}$$

and then derive a control law for the entire system (1.1) using backstepping [3]. In [5], Teel used the nested saturation control law

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$$v(x) = -\sigma_3 \left( y_3 + \sigma_2 \left( y_2 + \sigma_1 \left( y_1 \right) \right) \right),$$

where

$$\begin{aligned} y_1 &= -x_1 - 2x_2 + x_3 \\ y_2 &= -x_2 + x_3 \\ y_3 &= x_3, \end{aligned} \tag{2.2}$$

and

$$\sigma_i(s) = \begin{cases} -M_i, \text{ if } s < -M_i \\ \quad s \text{ , if } |s| \leq M_i \\ M_i, \text{ if } s > M_i \text{ ; } i = 1, 2, 3. \end{cases} \tag{2.3}$$

With $M_3$, the magnitude of the perturbation $x_1 v^2$ in (2.1) can be rendered arbitrarily small in any compact set. In [6], Teel proved the *semiglobal* result: by reducing the saturation level $M_3$, the region of attraction can be made arbitrarily large.

To achieve *global* stabilization of (2.1), we let the saturation levels depend on a function $\alpha(x)$. With this function, we attenuate the destabilizing effect of $x_1 v^2$ in larger regions. However, the stability analysis is more complicated than in the case of constant saturation levels. The state-dependent saturation levels must be large enough, but such that the solutions eventually enter a region in which the control law is not saturated.

**Theorem 1** *The equilibrium $x = 0$ of the system (2.1) is made globally asymptotically stable and locally exponentially stable by the control law:*

$$v(x) = -\sigma_3(x_3) - \alpha(x)\sigma_2 \left[ \frac{x_3 - x_2}{\alpha(x)} + \sigma_1 \left( \frac{x_3 - x_1 - 2x_2}{\alpha(x)} \right) \right], \tag{2.4}$$

*with*

$$\alpha(x) = \frac{1}{\sqrt{1 + x_1^2 + x_2^2}},$$

*and with the constraints on $M_3 > M_2 > M_1 > 0$ specified in the proof.*

**Proof:** We first show a property of the function $\alpha(x)$. Its time derivative along the solutions of (2.1) is

$$\dot{\alpha}(x(t)) = \frac{x_1 x_2 + x_2(-\sin x_3 + x_1 v^2)}{1 + x_1^2 + x_2^2} \alpha(x(t)).$$

In view of the inequalities

$$\begin{aligned} &|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2), \\ &|x_2 \sin x_3| \leq |x_2| \, |x_3|, \\ &|x_1 x_2 v^2| \leq \frac{v^2}{2}(x_1^2 + x_2^2) \leq \frac{1}{2}(M_3 + M_2)^2 (x_1^2 + x_2^2) \end{aligned}$$

the estimate

$$|\dot{\alpha}(x(t))| \leq \mu \alpha(x(t)), \tag{2.5}$$

with $\mu = \frac{1}{2} + \frac{M_3}{2} + \frac{(M_3 + M_2)^2}{2}$, holds whenever $|x_3| \leq M_3$. Thus, if $M_3 + (M_2 + M_3)^2 < \frac{1}{2}$, then $\frac{1}{2} < \mu < \frac{3}{4}$. In the following, to simplify the notation, we will use the change of variables (2.2).

Because the control law (2.4) is bounded, the solutions of the closed-loop system exist for all $t \geq 0$. We first show that there exists a finite time $T_3 > 0$, such that

$$\begin{aligned} |y_3(t)| &\leq c_1 M_2 \alpha(x(t)), \quad \text{and} \\ |v(x(t))| &\leq (1 + c_1) M_2 \alpha(x(t)), \quad \forall t \geq T_3, \end{aligned} \tag{2.6}$$

for some constant $c_1$, independent of $M_2$. (Henceforth, $c_i$ will denote a positive constant). In view of (2.4), the bound for $v$ follows from the bound for $y_3$. The time derivative of $V_3 = \frac{y_3^2}{2}$ is

$$\begin{aligned} \dot{V}_3 = y_3 \dot{y}_3 = -\sigma_3(y_3) y_3 - \alpha(x)\sigma_2(x) y_3 \leq \\ -|y_3| \Big( |\sigma_3(y_3)| - M_2 \Big), \end{aligned}$$

which proves that $|y_3(t)| \leq M_3$, after a finite time. Therefore, without loss of generality, we assume that $|y_3(0)| \leq M_3$, which yields

$$y_3(t) = y_3(0) e^{-t} - e^{-t} \int_0^t e^s \alpha(x(s)) \sigma_2(x(s)) \mathrm{d}s. \tag{2.7}$$

It follows from the bounds on $\mu$ that $|y_3(0)| e^{-t} \leq M_2 \alpha(x(0)) e^{-\mu t}$, for $t > T_3$ large enough, and, from (2.5), that $\alpha(x(0)) e^{-\mu t} \leq \alpha(x(t))$. Hence,

$$|y_3(0)| e^{-t} \leq M_2 \alpha(x(t)),$$

which bounds the first term in (2.7). To bound the second term, we integrate by parts:

$$\begin{aligned} &\left| e^{-t} \int_0^t e^s \alpha(x(s)) \sigma_2(x(s)) \mathrm{d}s \right| \leq \\ &M_2 e^{-t} \int_0^t e^s \alpha(x(s)) \mathrm{d}s = M_2 e^{-t} \Bigg( [e^t \alpha(x(t)) \\ &- \alpha(x(0))] - \int_0^t e^s \dot{\alpha}(x(s)) \mathrm{d}s \Bigg) \leq M_2 \alpha(x(t)) \\ &- M_2 e^{-t} \alpha(x(0)) + \mu M_2 e^{-t} \int_0^t e^s \alpha(x(s)) \mathrm{d}s \\ &\leq M_2 \alpha(x(t)) + \mu M_2 e^{-t} \int_0^t e^s \alpha(x(s)) \mathrm{d}s, \end{aligned}$$

and obtain

$$\left| e^{-t} \int_0^t e^s \alpha(x(s)) \sigma_2(x(s)) \mathrm{d}s \right| \leq$$
$$M_2 e^{-t} \int_0^t e^s \alpha(x(s)) \mathrm{d}s \leq \frac{M_2}{(1-\mu)} \alpha(x(t)),$$

which completes the proof of the bounds (2.6).

Using the change of variables (2.2), the second equation of (2.1), for $t \geq T_3$, is

$$\dot{y}_2 = -\alpha(x)\sigma_2 \left[ \frac{y_2}{\alpha(x)} + \sigma_1 \right] + (\sin y_3 - y_3) - x_1 v^2.$$

From the bounds (2.6), we deduce that $|(\sin y_3 - y_3) - x_1 v^2| \leq c_2 M_2^2 \alpha(x(t))$, for all $t \geq T_3$, and for some $c_2$, independent of $M_2$. We choose $M_2$ such that $|(\sin y_3 - y_3) - x_1 v^2| \leq \frac{1}{100} M_2 \alpha(x)$. In view of (2.6), this can always be achieved with a small enough $M_2$.
The time derivative of $V_2 = \frac{1}{2} y_2^2$ satisfies

$$\dot{V}_2 \leq -\frac{99}{100} M_2 |y_2| \alpha(x), \text{ for}$$
$$|y_2 + \alpha(x)\sigma_1| \geq M_2 \alpha(x). \qquad (2.8)$$

It follows that $y_2(t)$ is bounded, and so is $\dot{x}_1(t) = x_2(t)$. This means that $x_1(t)$ does not grow faster than a linear function of time, which implies that

$$\lim_{s\to\infty} \int_{T_3}^{\infty} \alpha(x(s)) \mathrm{d}s =$$
$$\lim_{s\to\infty} \int_{T_3}^{\infty} \frac{1}{\sqrt{1 + x_1^2(s) + x_2^2(s)}} \mathrm{d}s \to \infty. \qquad (2.9)$$

Let $T_2 = \inf\{t \geq T_3 : |y_2(t) + \alpha(x(t)) \ \sigma_1(x(t))| \leq M_2 \alpha(x(t))\}$. Because $V_2(y_2(t))$ cannot be negative, we conclude from (2.8) and (2.9) that $T_2$ is finite.
Next, we show that

$$\left| \frac{y_2}{\alpha(x)} + \sigma_1 \right| \leq M_2, \text{ for all } t \geq T_2. \qquad (2.10)$$

Introducing $z_2 \triangleq |y_2 + \alpha(x)\sigma_1| - M_2\alpha(x)$, we only need to prove that, whenever $z_2(t^*) = 0$, then $\dot{z}_2(t^*) < 0$. An explicit derivation yields

$$\dot{z}_2(t^*) \leq -\frac{7}{8} M_2 \alpha(x) + \mu M_2 \alpha(x) + \left| \frac{d}{dt}(\alpha(x)\sigma_1(x)) \right|,$$

$$\left| \frac{d}{dt}(\alpha(x)\sigma_1(x)) \right| \leq (1+2\mu) M_1 \alpha(x) + |(\sin y_3 - y_3) - x_1 v^2|.$$

We conclude that $\dot{z}_2(t^*) < 0$, provided that $M_2$ is sufficiently small, and that $M_1 < c_3 M_2$, where $c_3$ is a constant independent of $M_2$.

Using again the change of variables (2.2), we deduce from (2.10) that the first equation of (2.1), for $t \geq T_2$, is

$$\dot{y}_1 = -\alpha(x)\sigma_1 \left[ \frac{y_1}{\alpha(x)} \right] + 2(\sin y_3 - y_3) - 2x_1 v^2$$

With $c_4 > 0$, independent of $M_2$, we select $M_1$ in the interval

$$c_4 M_2 < M_1 < c_3 M_2, \qquad (2.11)$$

so that, for $M_2$ sufficiently small, $2|(\sin y_3 - y_3) - 2x_1 v^2| < \frac{4}{17} M_1 \alpha(x)$. (A detailed computation of all the constants $c_i$ can be found in [7])
The time derivative of $V_1 = \frac{1}{2} y_1^2$ satisfies

$$\dot{V}_1 \leq -\frac{13}{17} M_1 |y_1| \alpha(x), \text{ for } |y_1| \geq M_1 \alpha(x).$$

Using the same argument as for $y_2$, we conclude that there exists a finite time $T_1 \geq T_2$, such that $|y_1| \leq M_1 \alpha(x)$. To show that $\left| \frac{y_1}{\alpha(x)} \right| \leq M_1$, for $t \geq T_1$, we introduce a new variable $z_1 \triangleq |y_1| - M_1 \alpha(x)$. Then $z_1(t^*) = 0$ implies $\dot{z}_1(t^*) \leq -(\frac{13}{17} - \mu) M_1 \alpha(x) < 0$, which proves that $z_1(t) \leq 0$, for all $t \geq T_1$.

For $t \geq T_1$, the control law is no longer saturated,

$$v = -y_1 - y_2 - y_3,$$

and the closed loop system becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} +$$
$$\begin{bmatrix} 2(\sin y_3 - y_3) - 2x_1(y_1 + y_2 + y_3)^2 \\ (\sin y_3 - y_3) - x_1(y_1 + y_2 + y_3)^2 \\ 0 \end{bmatrix}. \qquad (2.12)$$

Because $A$ is Hurwitz, the equilibrium $y = 0$ of (2.12) is locally exponentially stable. Then $y = 0$ is globally asymptotically stable and locally exponentially stable, provided that $M_2$ is chosen sufficiently small, and $M_1$ is selected according to (2.11). ■

### 3 The Ball–and–Beam System

Only a slight modification of the proof shows that we can use a smooth saturation function instead of (2.3). Once such a smooth state feedback control law is designed, global stabilization of the ball-and-beam system proceeds by backstepping. With a virtual control law $x_4 = v(x_1, x_2, x_3)$, we stabilize the augmented system (1.1). The change of coordinates $z_4 = x_4 - v(x_1, x_2, x_3)$ yields

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_3 + x_1\left(z_4 + v(x_1,x_2,x_3)\right)^2\\ \dot{x}_3 &= z_4 + v(x_1,x_2,x_3)\\ \dot{z}_4 &= u - \dot{v}(x_1,x_2,x_3,x_4),\end{aligned}$$

where $(x_1,x_2,x_3,z_4) \in \mathbb{R} \times \mathbb{R} \times (-\frac{\pi}{2},\frac{\pi}{2}) \times \mathbb{R}$.

The time derivative of $V_{34} = \frac{1}{2}z_4^2 + \int_0^{x_3} \tan(s)ds$ is

$$\dot{V}_{34} = z_4(u-\dot{v}) + (z_4+v)\tan x_3.$$

We design the control law

$$u = -kz_4 - \tan x_3 + \dot{v}(x_1,x_2,x_3,x_4), \quad k>0, \quad (3.1)$$

where $\dot{v}(x_1,x_2,x_3,x_4)$ is known from the expression for $\frac{\partial v}{\partial x_i}$ and $\dot{x}_i$ in (1.1). With (3.1), we obtain

$$\dot{V}_{34} = -kz_4^2 + v(x_1,x_2,x_3)\tan x_3,$$

so, after a finite time,

$$\dot{V}_{34} \le -c_5 V_{34} + \alpha(x)|\sigma_2|, \quad (3.2)$$

for some $c_5$. The proof is then pursued as in Theorem 1, using the estimate (3.2) to replace the explicit solution (2.7).

## 4 Simulations

To evaluate the control law proposed in (3.1), we have performed a set of simulations using data made available to us by Professor T. L. Vincent, from his laboratory experiment at the University of Arizona [8]. The radius of the ball is R=.0012m, its density is $\rho$=7700$\frac{kg}{m^3}$. The beam is driven by a D.C. motor, with the armature resistance $R_A$=7.7Ω, the torque motor constant $k_\Gamma$=0.263$\frac{Nm}{A}$ and the back emf coefficient $k_\beta$=0.263$\frac{Vs}{rad}$. The moment of inertia of the beam and the motor is $J+J_M$=0.119 kgm$^2$.

As a smooth version of (2.4), we replace the saturation functions with *arctan(s)* having unitary slope around origin. Taking into account the physical coefficients, the virtual control law now is

$$v(x) = -2y_3 - \frac{18}{\pi}\alpha(x)\arctan\frac{\pi}{18}\left[\frac{y_2}{\alpha(x)} + \frac{20}{\pi}\arctan\frac{\pi}{20}\left[\frac{5y_1}{\alpha(x)}\right]\right] \quad (4.1)$$

The gains in (3.1) and (4.1) were tuned to improve the transients. One way to judge the performance improvement achieved with the designed nonlinear controller is to compare it with the performance of a linear controller [8] in the range where the linear design also achieves stability. This is done with plots in Figures 1, 2 and 3.

As shown in Fig 1, during an initial swing of the ball, the beam's angle and velocity are driven close to zero. After that, the ball's velocity and position converge to the equilibrium. This behavior is superior to the linear design in Fig 2, which is more oscillatory. It is significant that this performance improvement is achieved without an increase in control effort, shown in Fig 3.

## References


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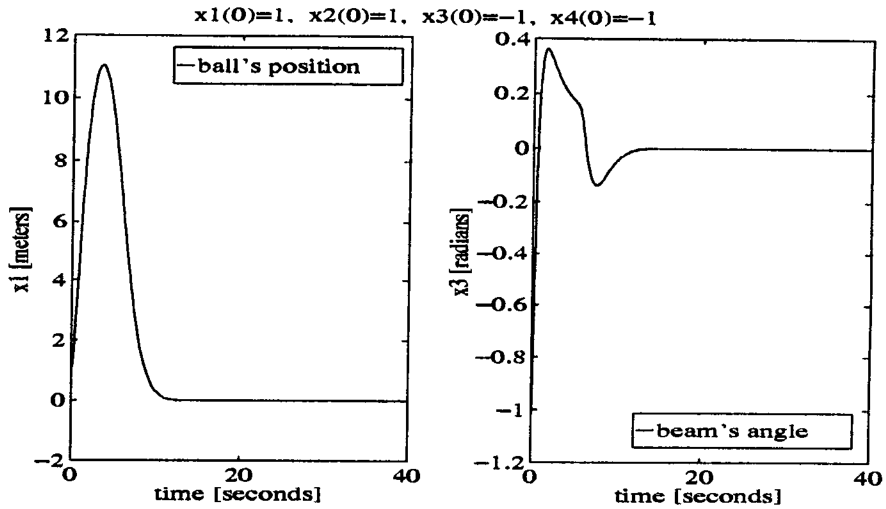


**Figure 1:** Nonlinear Controller

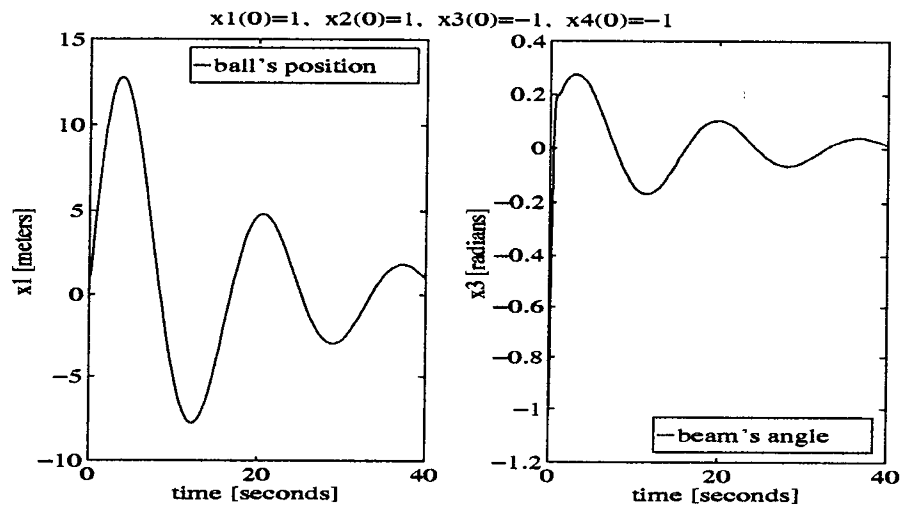


**Figure 2:** Linear Controller

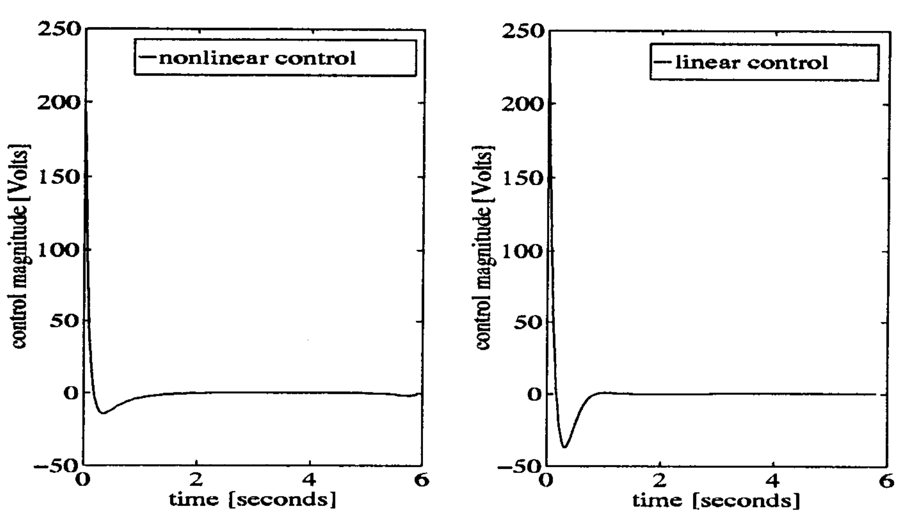


**Figure 3:** Control Effort