Nonlinear Control System Design via Dynamic Order Reduction

Reza Ghanadan

CCEC-94-0301

V. Approximate Tracking in the VTOL Aircraft System

An application of the control methodology described in this paper is in the aircraft control design. The dynamics of an aircraft exhibits two timescale behavior. The aircraft translational dynamics is in slow timescale while the rotational dynamics are in much faster timescale. A full aircraft model has 12 state variables involving the position and attitude dynamics [1, 12]. When the objective is to control the position of the aircraft during the vertical takeoff and landing, the dynamics exhibit nonminimum phase behavior. However, using dynamic order reduction techniques, we not only remove the difficulty of dealing with a nonminimum phase system, as described in Section III, but also compared with dynamic extension schemes the control law will be of lower order with fewer fast control loops to implement. Moreover, the overall feedback linearization procedure is simplified.

For the sake of illustration, we apply this design scheme to the VTOL aircraft control problem studied in [4], where an approximate decoupling controller was proposed. The equations of motion for this aircraft are given as:

$$\begin{aligned} \ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\ \ddot{y} &= u_1 \cos\theta + \epsilon u_2 \sin\theta + g \\ \ddot{\theta} &= u_2 \end{aligned} \tag{5.1}$$

where outputs x and y give the position of the aircraft center of mass, θ is the angle of the aircraft relative to the x-axis, u_1 and u_2 are the thrust and the rolling moment, g is the gravitational acceleration normalized to -1, and ϵ is a small coupling coefficient between the rolling moment u_2 and the lateral acceleration of the aircraft $\ddot{x}$ and $\ddot{y}$. The objective is to find a stable feedback law that decouples outputs x and y. We require x to track a smooth trajectory $x_r(t)$ while y remains at zero.

System (5.1) is an example of a slightly non-minimum phase system where the regular perturbation causes a nearly singular decoupling matrix and unstable zero dynamics. However, it can be shown that the unperturbed system ($\epsilon = 0$) has vector relative degree $(4, 4)$ under a dynamic feedback control and is a minimum phase system.

Since one of the important design objectives here is that the altitude y remains at zero, we first decouple the altitude dynamics to achieve *exact* control for tracking and regulation in the

y direction. This task can simply be performed using the following input transformation before splitting the dynamics into two timescales:

$$u_1 = \frac{1}{\cos\theta}[1+\omega_2] - \epsilon \tan\theta u_2 \tag{5.2}$$

The resulting equations describing this system are now given by:

$$\begin{aligned} \ddot{x} &= -\tan\theta\,(1+\omega_2) + \epsilon\tfrac{u_2}{\cos\theta} \\ \ddot{y} &= \omega_2 \\ \ddot{\theta} &= u_2 \end{aligned} \tag{5.3}$$

Following the procedures in Section II, we define the new coordinates from (2.5) and (2.7) as: $x_i = x_i, y_i = y_i, z_1 = \theta, z_2 = \frac{1}{k}\dot{\theta}$. In these new coordinates, (5.1) and (5.6) can be rewritten as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tan z_1\,(1+\omega_2) + \epsilon\tfrac{u_2}{\cos\theta} \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \omega_2 \\ \tfrac{1}{k}\dot{z}_1 &= z_2 \\ \tfrac{1}{k}\dot{z}_2 &= \tfrac{1}{k^2}u_2 \end{aligned} \tag{5.4}$$

which is in the general form of the set of equations (2.8)-(2.10). We can design k to be large, but such that $k\epsilon^2$ is at most $O(1)$. Following our analysis in the last section, equations (2.4) and (2.7), we have $d=2, \gamma=5$. Therefore, if an approximate value for ϵ is known, as in (2.4) we may simply rescale ϵ such that:

$$\frac{1}{k^\gamma} = \epsilon$$

which can be solved for k: $\frac{1}{k} = \epsilon^{0.2}$. The control law is readily given by (3.3) with $\epsilon_2 = \epsilon^d = \frac{1}{k^2}$:

$$u_2 = -k^2(\alpha z_1 + \beta z_2 - v_2) \tag{5.5}$$

where α and β are such that $s^2 + \beta s + \alpha$ is Hurwitz. In the original coordinates, control law (5.5) is represented by:

$$u_2 = -k^2\alpha\theta - \beta k\dot{\theta} + k^2 v_2 \tag{5.6}$$

and the resulting θ-dynamics is given by:

$$\frac{1}{k^2}\ddot{\theta} + \frac{\beta}{k}\dot{\theta} + \alpha\theta = v_2 \tag{5.7}$$

The overall system can now be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tan z_1\,(1+\omega_2) + \epsilon\tfrac{u_2}{\cos\theta} \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \omega_2 \\ \tfrac{1}{k}\dot{z}_1 &= z_2 \\ \tfrac{1}{k}\dot{z}_2 &= -\alpha z_1 - \beta z_2 + v_2 \end{aligned} \tag{5.8}$$

Reduced Order Model: The reduced order system is obtained by formally setting $\epsilon = 0$, and hence $\frac{1}{k} = 0$ in (5.8), to solve for the steady-state values of z_1 and z_2:

$$z_{2s}^0 = 0, z_{1s}^0 = \frac{v_2}{\alpha} \quad \rightarrow H^0(t, x, \dot{x}, y, \dot{y}) = \begin{bmatrix} \frac{v_2^0}{\alpha} \\ 0 \end{bmatrix} \tag{5.9}$$

Using these values in (5.3), the zero order reduced system is:

$$\begin{aligned} \ddot{x}_s &= -(1+\omega_2)\tan\left(\frac{v_2^0}{\alpha}\right) \\ \ddot{y}_s &= \omega_2 \end{aligned} \tag{5.10}$$

v_2^0 can then be designed for linearizing x-dynamics by setting $\ddot{x}_s = \omega_1$ to obtain:

$$\begin{aligned} v_2^0 &= -\alpha \arctan\left(\frac{\omega_1}{1+\omega_2}\right) \\ \omega_1 &= \ddot{x}_r - b_{12}(\dot{x}-\dot{x}_r) - b_{11}(x-x_r) \\ \omega_2 &= -b_{22}\dot{y} - b_{21}y \end{aligned} \tag{5.11}$$

The resulting reduced order model under above feedbacks becomes:

$$\begin{aligned} \ddot{e}_{xs} + b_{12}\dot{e}_{xs} + b_{11}e_{xs} = 0 \\ \ddot{y}_s + b_{22}\dot{y}_s + b_{21}y_s = 0 \end{aligned} \tag{5.12}$$

which is exponentially stable in $(e_{xs}, \dot{e}_{xs}, y_s, \dot{y}_s)$.

Boundary-Layer-System:

Define: $\zeta = z - H^0(t, x, \dot{x}, y, \dot{y}), \tau = kt$, then the boundary-layer-system is given by:

$$\frac{d\zeta}{d\tau} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \zeta \tag{5.13}$$

which is exponentially stable uniformly in $(x, \dot{x}, y, \dot{y})$.

Figure (1), shows the outputs (x, y), angle θ, tracking error, and the deviation of z_i^0 from z_i for this example subject to control laws (5.2), (5.6), and (5.11), with $\epsilon = 0.1, \frac{1}{k} = \epsilon^{0.2}$. We next illustrate the performance of this system when first order correction term is added to the control. Let's rewrite (5.4) in the general form:

$$\begin{aligned} \dot{X} &= \bar{f}(X, Z, \mu) \\ \mu\dot{Z} &= \bar{g}(X, Z) \end{aligned} \tag{5.14}$$

where $\mu \triangleq \frac{1}{k}$, $X = (x, \dot{x}, y, \dot{y})$, and $Z = (\theta, \mu\dot{\theta})$. The slow manifold M_μ of (5.1) is of the form:

$$M_\mu : \quad Z = H(t, X, \mu) \tag{5.15}$$

and for this to be an invariant manifold for the (5.14) it must be true that:

$$\bar{g}(X, H(t, X, \mu)) = \mu \frac{\partial H(t, X, \mu)}{\partial X} \bar{f}(X, H(t, X, \mu)) + \mu \frac{\partial H(t, X, \mu)}{\partial t} \tag{5.16}$$

which $H(t, X, \mu)$ must satisfy for all X in the region of interest and all $\mu \in [0, \mu_0]$. Corresponding to the slow manifold (5.15), there exists a slow model:

$$\dot{X} = \bar{f}(X, H(t, X, \mu), \mu) \tag{5.17}$$

which describes the behavior of X on M_ϵ. (5.16) is known as the *manifold condition* [11]. To solve equation (5.16) for $H(t, X, \mu)$ approximately, we start with the Taylor expansion of the slow manifold:

$$Z = H(t, X, \mu) = H^0(t, X) + \mu H^1(t, X) + \mu^2 H^2(t, X) + \cdots \tag{5.18}$$

Then, we substitute (5.18) in the manifold condition (5.16) and evaluate H^i by recursive algebraic operations.

The zero order term is given by setting $\mu = 0$ in (5.16) which results in the zero order reduced model as before, i.e. $\bar{g}(X, H^0(t,X)) = 0$, $Z^0 = H^0(t,X)$, with equations (5.9) and (5.10). The first order correction term for tracking is given by:

$$H^1(t,X) = \left(\frac{\partial \bar{g}(X, H^0(t,X))}{\partial Z}\right)^{-1} \left[\frac{H^0(t,X)}{\partial X}\bar{f}(X, H^0(t,X), 0) + \frac{H^0(t,X)}{\partial t}\right] \tag{5.19}$$

which is well defined for this example since the Jacobian matrix $\frac{\partial \bar{g}(X,H^0(t,X))}{\partial Z}$ is nonsingular and we have:

$$\left(\frac{\partial \bar{g}(X, H^0(t,X))}{\partial Z}\right)^{-1} = \begin{pmatrix} -\frac{\beta}{\alpha} & -\frac{1}{\alpha} \\ 1 & 0 \end{pmatrix} \tag{5.20}$$

$$\frac{H^0(t,X)}{\partial X} = \frac{1}{\alpha}\begin{bmatrix} \frac{\partial v_2}{\partial X} \\ 0 \end{bmatrix} = \frac{1}{(1+\omega_2)^2 + \omega_1^2}\begin{bmatrix} b_{11}(1+\omega_2) & b_{12}(1+\omega_2) & -b_{21}\omega_1 & -b_{22}\omega_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first order reduced system is obtained by setting $z_1 = z_1^1, z_2 = z_2^1$, and $\mu^i = \frac{1}{k^i} = 0 \ \ \forall i \geq 2$ in (5.8), or directly from (5.3), by setting $\theta = z_1^1$ and $\epsilon = 0$ (since $\epsilon = \frac{1}{k^3}$):

$$\begin{aligned} \ddot{x} &= -(1+\omega_2)\tan(z_1^1) \\ \ddot{y} &= \omega_2 \end{aligned} \tag{5.21}$$

where $z_1^1 = H^0(t,X) + \mu H^1(t,X)$. Hence, the first order corrected control for v_2 is:

$$v_2 = -\alpha \arctan\left[\frac{\omega_1}{1+\omega_2}\right] - \frac{\alpha}{k}H_1^1(t,X) \tag{5.22}$$

with u_1 and u_2 given as in (5.2) and (5.6). Figure (2) plots the response of the VTOL system subject to the first order corrected control law. Note that compared with the zero order control, the performance is improved when the correction term is included in the control design.

VI. Conclusion

In this paper we have presented an approach for the *approximate* tracking of nonlinear systems. We showed that applications of singular perturbation theory simplifies the input-output linearizing transformation in nonlinear systems by identifying and eliminating the negligible (fast) state variables, and converting a suitable state variable into a "control variable" that appears linearly in the slow "dominant" dynamics. The design is carried on based on the resulting reduced order model and is such that, in the steady state, the "control like" state is forced to approach the desired tracking control law for the reduced order system. Because the exact size of perturbation in the system is usually unknown, we used an estimate of its bound in the controller design to show the stability of the closed loop system and guaranteed bounds on the tracking error. Since the normal form of a nonlinear system is not robust to regular perturbations, our design approach is particularly useful for systems with high order relative degree subject to perturbations and for slightly nonminimum phase systems. Also, since systems with high order dynamics are more difficult to control and fast high order control loops are often hard to implement, splitting the control loop into the lower order fast and slow loops using dynamic order reduction techniques makes the design easier to implement.

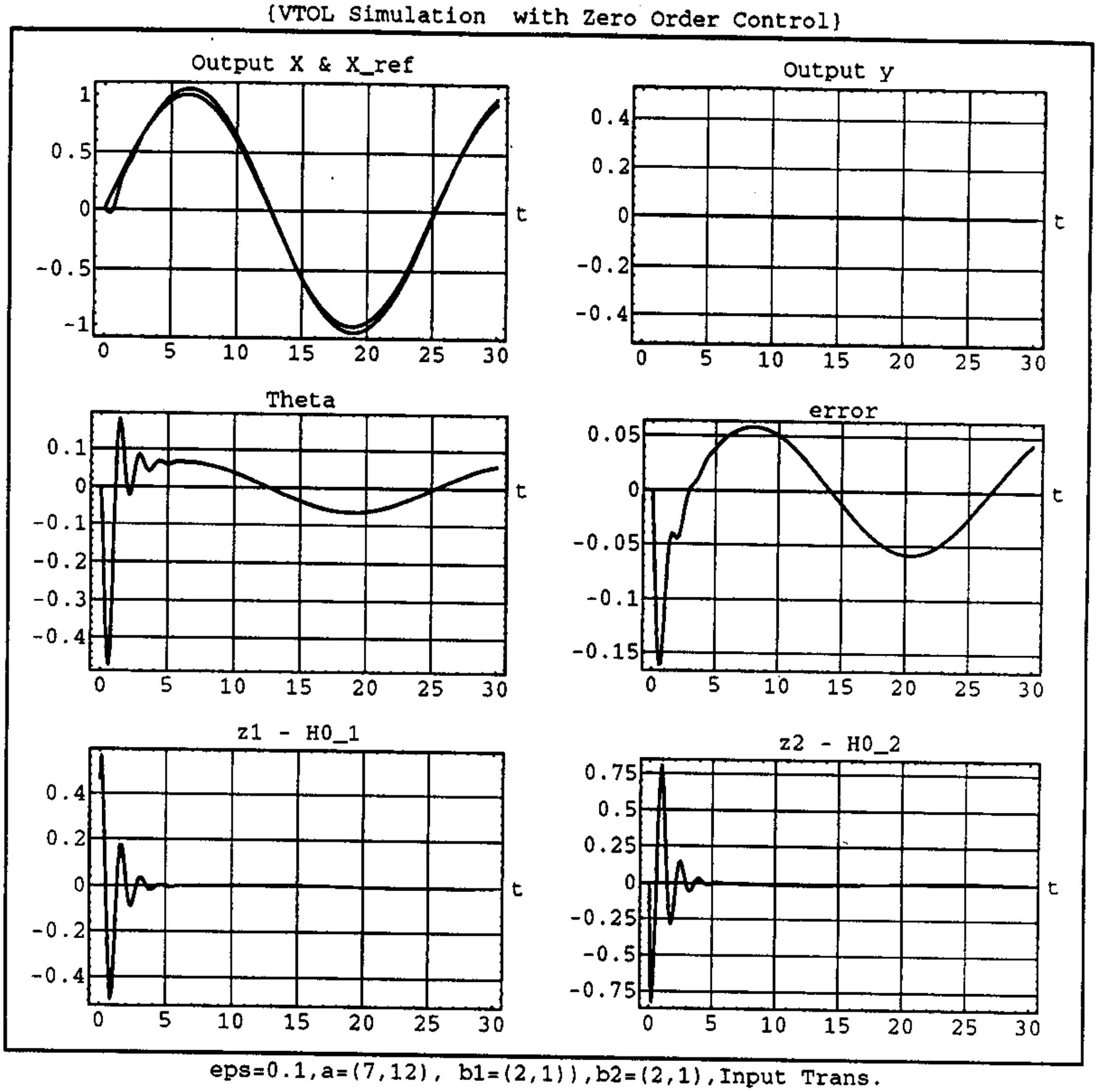

Figure 1: Zero Order Control

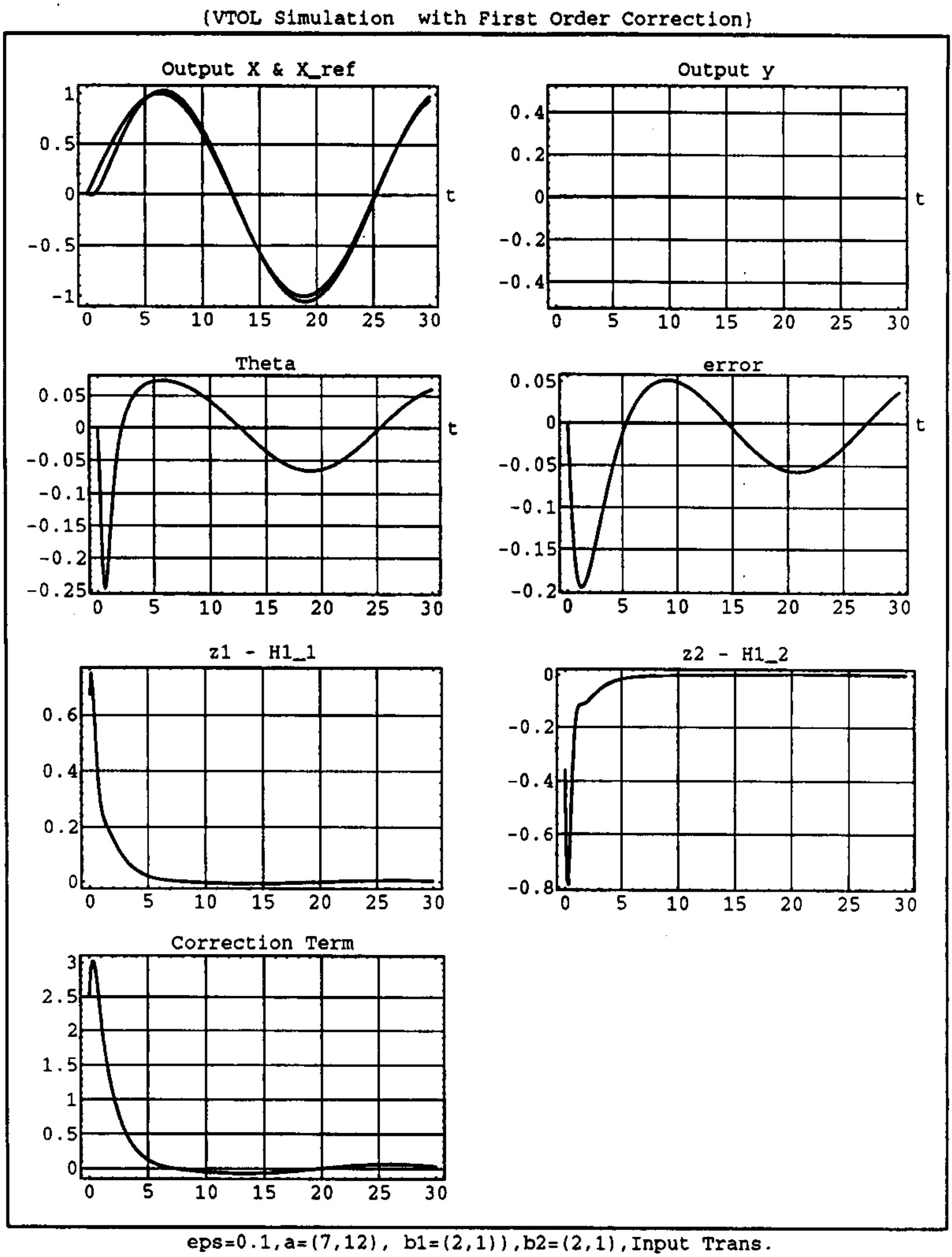

Figure 2: First Order Control