TORA Example: Cascade- and Passivity-Based Control Designs

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Abstract—**We consider the problem of feedback stabilization of translational oscillations by a rotational actuator (TORA) system. The main obstacle to controller design is nonlinear coupling from the rotational to the translational motion through a sinusoidal term. We present several controller designs based on the cascade and passivity paradigms.**

I. Introduction

In this paper we consider the problem of controlling translational oscillations with a rotational actuator (TORA), brought to our attention by Bernstein [1], [2]. The motivation comes from possible use of rotational actuation to remove some limitations of the current proof-mass actuators. The problem is also of interest as a case study in nonlinear controller design because the model exhibits nonlinear interaction between the translational and rotational motions.

The control laws designed in this paper are of two types: cascade controllers and feedback passivating controllers. Their properties are illustrated with simulations.

II. TORA System Model

The structure of the TORA system depicted in Fig. 1 corresponds to the physical system which has been built by Bernstein and co-workers at the University of Michigan. It consists of a platform that can oscillate without damping in the horizontal plane (no gravity effect). On the platform a rotating eccentric mass is actuated by a dc motor. Its motion applies a force to the platform which can be used to damp the translational oscillations. Assuming that the motor torque is the control variable, our task is to find a control law to asymptotically stabilize the system at a desired equilibrium.

We let x_1 be the normalized displacement of the platform from the equilibrium position, $x_2 = \dot{x}_1$, $x_3 = \theta$ be the angle of the rotor, and $x_4 = \dot{x}_3$. In these coordinates the dynamics of the system are described by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

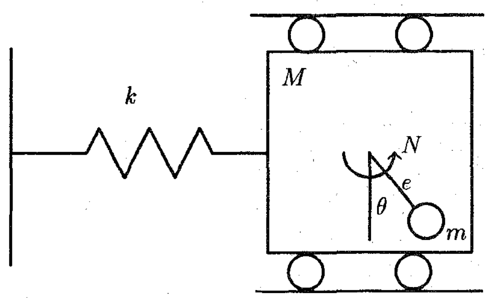

Fig. 1. TORA system configuration.

where u is the torque applied to the eccentric mass. The vectors f and g are given by

$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ x_4 \\ \dfrac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \dfrac{1}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix}$$

with ε being a constant parameter which depends on the rotor, platform masses and eccentricity (a typical value $\varepsilon = 0.1$ has been used in simulations). All quantities are normalized in dimensionless units. More details about the derivation of the model can be found in [1] where the following new state variables are introduced:

$$\begin{aligned} z_1 &= x_1 + \varepsilon \sin x_3 \\ z_2 &= x_2 + \varepsilon x_4 \cos x_3 \\ y_1 &= x_3 \\ y_2 &= x_4 \end{aligned} \tag{2}$$

and the feedback transformation

$$\begin{aligned} v &= \frac{1}{1 - \varepsilon^2 \cos^2 y_1} \{\varepsilon \cos y_1 [z_1 - (1 + y_2^2)\varepsilon \sin y_1] + u\} \\ &\triangleq \alpha(z_1, y_1) + \beta(y_1) u \end{aligned} \tag{3}$$

is employed to bring the system into the simpler form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \varepsilon \sin y_1 \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= v. \end{aligned} \tag{4}$$

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Because ε is always smaller than one, the transformation from u to v is nonsingular.

III. Cascade System Designs

An attempt to design a controller by feedback linearization fails because, after the required change of variables, the input to the system is multiplied by $\cos y_1$. This introduces a singularity at $y_1 = \pm(\pi/2)$ and disallows the rotor action outside these limits.

Instead of pursuing linearization, we pretend that y_1 is our control variable and design for it a control law $y_1 = y_1(z)$ that renders the origin for the z-subsystem globally asymptotically stable. One such control law is $y_1 = -\arctan(c_0 z_2)$ (cf. [1]). Recognizing that y_1 is not the control variable, and that the deviation from its desired value is $\xi_1 = y_1 + \arctan(c_0 z_2)$, we let $\xi_2 = \dot{\xi}_1$ and rewrite (4) as

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -z_1 + \varepsilon \sin[\xi_1 - \arctan(c_0 z_2)]\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= w.\end{aligned} \tag{5}$$

To linearize the ξ-subsystem we have defined w via one more feedback transformation: $w = v - [(2c_0^3 z_2)/(1+c_0^2 z_2^2)^2](-z_1 + \varepsilon \sin y_1)^2 + [c_0/(1+c_0^2 z_2^2)](-z_2 + \varepsilon y_2 \cos y_1)$.

The system (5) is in the cascade form

$$\begin{aligned}\dot{z} &= f(z, \xi)\\ \dot{\xi} &= A\xi + Bw.\end{aligned} \tag{6}$$

Its key property, not present in (4), is that $\dot{z} = f(z, 0)$ is globally asymptotically stable as one can verify by examining the derivative of the Lyapunov function $V_0 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$ along the trajectories of $\dot{z} = f(z, 0)$. This z-stability property allows us to apply the cascade design paradigm of [3] and [4].

(C1) Controller—Linear Cascade Control: The (C1) control law simply stabilizes the lower ξ-subsystem with ξ-feedback only. This is feasible because $\dot{z} = f(z, \xi)$ is globally Lipschitz, with the Lipschitz constant equal to one, so that any control law $w = -K\xi$ which gives all the eigenvalues of $A - BK$ with the real parts smaller than -1, makes the cascade system (6) globally asymptotically stable without the danger of peaking (see [4, Theorem 6.2]). Because of the particular structure of $f(z, \xi)$, using [5, Lemma 3.2], one can show that any globally asymptotically stabilizing feedback for the linear ξ-subsystem achieves global asymptotic stability of (6). While the cascade system (5) is stabilized by a linear ξ-feedback, the full (C1) feedback law $v = v_c$ for (4) is nonlinear

$$\begin{aligned}v_c = &-k_1[y_1 + \arctan(c_0 z_2)] - k_2 y_2\\ &- \frac{k_2 c_0}{1 + c_0^2 z_2^2}(-z_1 + \varepsilon \sin y_1)\\ &+ \frac{2c_0^3 z_2}{(1 + c_0^2 z^2)^2}(-z_1 + \varepsilon \sin y_1)^2\\ &- \frac{c_0}{1 + c_0^2 z_2^2}(-z_2 + \varepsilon y_2 \cos y_1).\end{aligned} \tag{C1}$$

The parameters $k_1 > 0$ and $k_2 > 0$ can be chosen to assign the desired dynamics to the ξ-subsystem. With the (C1) control the origin is a globally asymptotically stable equilibrium of the system (4).

(C2) Controller—Integrator Backstepping: Another approach to stabilize the cascade system (5) is integrator backstepping [3] also employed in [1]. The design steps given here are different, but the final control laws are the same.

Since the z-subsystem of (5) is globally asymptotically stable with the Lyapunov function $V_0 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$, we consider ξ_2 as our virtual control and try to make the derivative of $V_1(z, \xi_1) = p_0 V_0(z) + (p_1/2)\xi_1^2$ negative

$$\begin{aligned}\dot{V}_1 = &-p_0 \varepsilon z_2 \sin[\arctan(c_0 z_2)] + p_1 \xi_1 \xi_2\\ &+ p_0 \varepsilon z_2 \{\sin[\xi_1 - \arctan(c_0 z_2)] + \sin[\arctan(c_0 z_2)]\}.\end{aligned}$$

A choice of the virtual control resulting in $\dot{V}_1 \le 0$ is

$$\xi_{2d} = \begin{cases} -\varepsilon\left(\dfrac{p_0}{p_1}\right) z_2\{\sin[\xi_1 - \arctan(c_0 z_2)] \\ \quad + \sin[\arctan(c_0 z_2)]\}/\xi_1 - c_1\xi_1 & \text{if } \xi_1 \neq 0\\ -\varepsilon\left(\dfrac{p_0}{p_1}\right) z_2 \cos[\arctan(c_0 z_2)] & \text{if } \xi_1 = 0.\end{cases} \tag{7}$$

The smoothness of the control law (7) follows from $\sin(a - b) + \sin(b) = \sin a \cos b - \sin b\,(1 - \cos a)$ and the fact that $(\sin a)/a$ and $(1 - \cos a)/a$ are smooth functions.

In the next step we employ

$$V_2(z, \xi) = V_1(z, \xi_1) + \frac{p_2}{2}(\xi_2 - \xi_{2d})^2.$$

Our C2) controller is designed to achieve $\dot{V}_2 \le 0$. It is given by

$$\begin{aligned}v_b = &-c_2 y_2 - c_2 c_1[y_1 + \arctan(c_0 z_2)] + \frac{c_0 z_2}{1 + c_0^2 z_2^2}\\ &+ \left\{\varepsilon \frac{p_0 c_0}{p_1} z_2 \frac{\sin y_1 + \sin[\arctan(c_0 z_2)]}{(1 + c_0^2 z_2^2)\xi_1^2}\right.\\ &- \frac{c_0(c_1 + c_2)}{1 + c_0^2 z_2^2}\\ &- \varepsilon \frac{p_0}{p_1} \frac{\sin y_1 + \sin[\arctan(c_0 z_2)]}{\xi_1}\\ &+ \frac{2c_0^3 z_2(-z_1 + \varepsilon \sin y_1)}{(1 + c_0^2 z_2^2)^2}\\ &\left.- \varepsilon \frac{p_0}{p_1} \frac{c_0 z_2 \cos[\arctan(c_0 z_2)]}{(1 + c_0^2 z_2^2)\xi_1}\right\} \times (-z_1 + \varepsilon \sin y_1)\\ &- \frac{p_1}{p_2}[y_1 + \arctan(c_0 z_2)] - \varepsilon \frac{p_0}{p_1} z_2 y_2 \frac{\cos y_1}{\xi_1}\\ &+ \varepsilon \frac{p_0}{p_1} z_2 y_2 \frac{\sin y_1 + \sin[\arctan(c_0 z_2)]}{\xi_1^2} - c_1 y_2\\ &- c_2 \varepsilon \frac{p_0}{p_1} z_2 \frac{\sin y_1 + \sin[\arctan(c_0 z_2)]}{\xi_1}\\ &- \varepsilon c_0 y_2 \frac{\cos y_1}{1 + c_0^2 z_2^2}.\end{aligned} \tag{C2}$$

Although (C2) is significantly more complex than (C1), the backstepping approach has its own merits which we discuss next.

We first point out that (C1) can be obtained as a limiting case of (C2), because, if $k_1 = c_1 c_2 + (p_1/p_2)$, $k_2 = c_1 + c_2$, and $p_0 = \delta p_1$, $(\delta > 0)$, then

$$\lim_{\delta \to 0} v_b(z, y) = v_c(z, y).$$

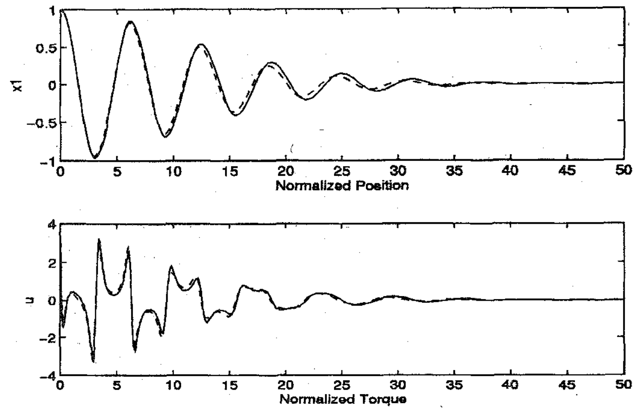

Fig. 2. Transient behavior with (C1) and (C2) control laws.

In other words, for some choices of the parameters, the two control laws would result in the same performance.

Indeed, simulations have shown that the two control laws are similar even beyond the limiting case. In Fig. 2 the plots[1] of translational displacement x_1 and the control effort u with (C1) (solid curve, $\delta = 0$) and with (C2) (dashed curve, $\delta = 10$) show a slight improvement with the backstepping method.

For a clearer illustration of some properties of the backstepping control we consider the following example:

$$\begin{aligned} \dot{x} &= -x + g(x)y \\ \dot{y} &= u. \end{aligned} \tag{8}$$

Since with $y \equiv 0$ the x-subsystem is globally asymptotically stable, in the second step of backstepping we let the Lyapunov function be

$$V = \frac{p_0}{2} x^2 + \frac{p_1}{2} y^2.$$

Its derivative along the trajectories of (8) is

$$\dot{V} = -p_0 x^2 + p_0 x g(x) y + p_1 y u$$

and a choice of u which makes $\dot{V}$ negative definite is

$$u = -\frac{p_0}{p_1} x g(x) - ay.$$

With this backstepping control the resulting closed-loop system is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & g(x) \\ -\frac{p_0}{p_1} g(x) & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This system has the form of a damped nonlinear oscillator. In contrast, a (C1)-type controller for (8) is simply $u = -ay$ and results in the closed-loop system which is upper triangular. Some of the benefits of the term $-(p_0/p_1)\, g(x)$ introduced by backstepping are:

1) If, for example, $g(x) = x^2$ a (C1)-type controller cannot guarantee global stability. This advantage is not present in the TORA system which is globally Lipschitz.
2) When $g(x)$ is globally Lipschitz, or even constant, say $g(x) = g_0$, then $-(p_0/p_1)\, g(x)$ improves the transient by reducing the overshoot of x due to an initial condition of y. If (p_0/p_1) is very small the overshoot is large and, vice-versa, if (p_0/p_1) is large the overshoot is small. For the TORA problem this improvement is negligible because the connection between the two subsystems is weak ($\varepsilon = 0.1$).

[1] All our plots show x_1 and u. To compare them with [1] and [2] the values shown should be divided by 10.

IV. Feedback Passivating Designs

The cascade designs in Section III resulted in complicated control laws that require full state feedback and cancellation of nonlinearities. Our more ambitious design objective is to use a reduced set of measurements and avoid cancellation of nonlinearities. This will simplify instrumentation and improve robustness.

A design paradigm which can be used to accomplish this objective is feedback passivation [6]. We design three passivating controllers: (P1), (P2), and (P3). Our basic passivating feedback controller (P1) employs a reduced number of states (z_1, y_1, y_2), but still cancels two nonlinear terms. To avoid cancellation, we refine the storage function and design the (P2) controller which has a surprisingly simple form $u = -k_1 y_1 - k_2 y_2$ and accomplishes our design objective. To increase the damping of x_1 beyond a limit imposed by the simple structure of (P2), we employ a storage function with increased penalty on z_1 and z_2. The resulting passivating controller (P3) achieves improved damping of x_1. It matches the damping obtained with the cascade control laws, but with smaller control effort.

We rewrite the TORA system in the (z, y)-coordinates with the original input u

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \varepsilon \sin y_1 \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \alpha(z_1, y_1) + \beta(y_1) u \end{aligned} \tag{9}$$

with α and β defined in (2).

(P1) Controller—Feedback Passivation: The (P1)-design illustrates the feedback passivation procedure. Its basis is the requirement that, for a system to be passive from an input u to an output $y = h(x)$, the relative degree one and weakly minimum phase conditions must be satisfied.

Choice of the Output Function: For the system (9) the constraint of relative degree one means that the output $h(z, y)$ must be a function of y_2. So, we let $h(z, y) = y_2$. With this output the corresponding zero dynamics subsystem

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \varepsilon \sin y_1 \\ \dot{y}_1 &= 0 \end{aligned} \tag{10}$$

is stable by inspection. This means that the system (9) with the output $y = y_2$ is weakly minimum phase.

Lyapunov Function for the Zero Dynamics: In this step we want to find a Lyapunov function $U(z, y_1)$ for (9) which will be incorporated into the passivating feedback law. Because y_1 is constant, we can treat (10) as a linear system and select

$$U(z, y_1) = \frac{1}{2}(z_1 - \varepsilon \sin y_1)^2 + \frac{1}{2} z_2^2 + \frac{k_1}{2} y_1^2 \tag{11}$$

with k_1 being a design parameter. The time derivative of U along the trajectories of the system (10) is nonpositive, in fact $\dot{U} = 0$, which is satisfactory, since (10) is not asymptotically stable.

Passivating Feedback and Storage Function: Note that the system (9) is already in the normal form as defined in [6]. Therefore, the feedback transformation

$$\begin{aligned} u &= \beta^{-1}\left(-\alpha - \frac{\partial U}{\partial y_1} + u_1\right) \\ &= \varepsilon^2 y_2^2 \sin y_1 \cos y_1 - \varepsilon^3 \cos^2 y_1 (z_1 - \varepsilon \sin y_1) \\ &\quad - (1 - \varepsilon^2 \cos^2 y_1)(k_1 y_1 - u_1) \end{aligned} \tag{12}$$

renders the system passive from the input u_1 to the output y_2 with respect to the storage function

$$\widetilde{W}(z, y) = \frac{1}{2}(z_1 - \varepsilon \sin y_1)^2 + \frac{1}{2} z_2^2 + \frac{k_1}{2} y_1^2 + \frac{1}{2} y_2^2. \tag{13}$$

Indeed, $\dot{\widetilde{W}} = y_2 u_1$ which means that the system (9), (12) is not only passive, but also lossless (see [6] and [7]).

A fundamental passivity property [8], [9] is that the feedback connection of a passive and a strictly passive block is stable in the sense that $y_2 \in \mathcal{L}_2$. Therefore, we can select any strictly passive feedback control law. The simplest one is $u_1 = -k_2 y_2$ with which we complete our (P1) controller design. Returning to the original control u, (P1) controller is

$$\begin{aligned} u &= \varepsilon^2 y_2^2 \sin y_1 \cos y_1 - \varepsilon^3 \cos^2 y_1 (z_1 - \varepsilon \sin y_1) \\ &\quad - (1 - \varepsilon^2 \cos^2 y_1)(k_1 y_1 + k_2 y_2). \end{aligned} \tag{P1}$$

Its passivity property guarantees only stability. Additional properties are needed to conclude asymptotic stability.

(P2) Controller—Passivation without Cancellation: Even though it is simpler than (C1) or (C2), the controller (P1) has not completely satisfied the design objective because it uses three states for feedback and cancels nonlinearities. We now show that, by modifying the storage function (13) as

$$\begin{aligned} W(z, y) &= \frac{1}{2}(z_1 - \varepsilon \sin y_1)^2 + \frac{1}{2} z_2^2 + \frac{k_1}{2} y_1^2 \\ &\quad + \tfrac{1}{2} y_2^2 (1 - \varepsilon^2 \cos^2 y_1) \end{aligned} \tag{14}$$

we can achieve the desired objective. The passivating feedback transformation with respect to W is $u = -k_1 y_1 + u_1$. This simple feedback achieves passivity from u_1 to y_2 since $\dot{W}(z, y) = y_2 u_1$.

The next step is to achieve global asymptotic stability. We close the feedback loop with the simplest strictly passive block: $u_1 = -k_2 y_2$. Thus our (P2) controller is

$$u = -k_1 y_1 - k_2 y_2. \tag{P2}$$

With the control law (P2) we have $\dot{W} = -k_2 y_2^2 \le 0$ which implies that the origin of the closed-loop system (9), (P2) is

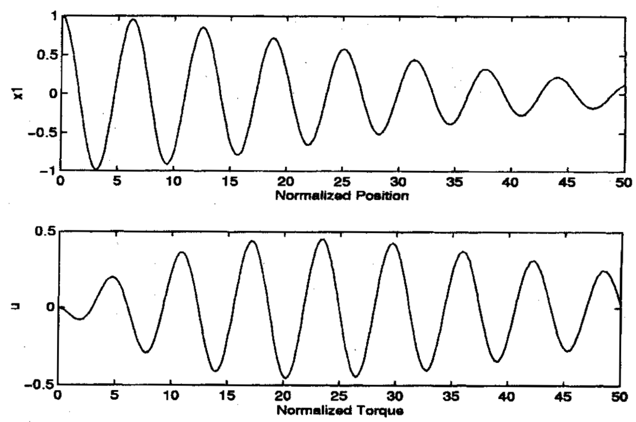

Fig. 3. Response of the (P2)-system.

stable in the sense of Lyapunov and all the states are bounded. Because W is radially unbounded these properties are global.

Now we employ LaSalle's invariance principle to prove asymptotic stability. The largest invariant set of the closed-loop system (9), (P2) contained in the set $\Omega = \{(z, y) \in R^4 \colon y_2 = 0\}$, where $\dot{W} = 0$, is determined from $\dot{y}_2 \equiv 0$, i.e.,

$$0 \equiv \varepsilon \cos y_1 (z_1 - \varepsilon \sin y_1) - k_1 y_1. \tag{15}$$

Because $\dot{y}_1 = y_2 \equiv 0$, $y_1 = const$. Then it follows from (15) that $z_1 = const \Rightarrow z_2 \equiv 0 \Rightarrow \dot{z}_2 = z_1 - \varepsilon \sin y_1 \equiv 0$. So the only solution of (15) inside the set Ω is $z_1 = y_1 = 0$ which proves the global asymptotic stability of $z_1 = z_2 = y_1 = y_2 = 0$.

A typical response of the (P2)-system is shown in Fig. 3 with $k_1 = 1$, $k_2 = 0.14$ chosen for the fastest possible convergence.

We remark that (P2) controller is a linear version of the "passive absorber" proposed in [2]. and the plots in Fig. 3 closely resemble those obtained in [2].

The (P2) controller achieves our design objectives because it does not cancel nonlinearities and uses only the rotor variables y_1 and y_2 for feedback. The simulations show that its control effort is smaller than with (C1) and (C2). A drawback of (P2), however, is that the speed of convergence of x_1 to zero cannot be increased by adjusting the gains. This problem will be resolved by our (P3) controller in the next section. But first we illustrate some of the possibilities of achieving different control objectives with simple (P2)-type control laws.

(P2) with Bounded Control: Here we design a control law which achieves global asymptotic stabilization with the norm of the input never exceeding some prescribed value $\delta > 0$. One such control law is

$$u_b = -k_1 \frac{y_1}{\sqrt{1 + y_1^2}} - (\delta - k_1) \arctan y_2 \tag{16}$$

with $k_1 < \delta$. The magnitude of the input u_b is obviously bounded by δ and global asymptotic stability follows from the Lyapunov function

$$W_b = W - \frac{k_1}{2} y_1^2 + k_1\left(\sqrt{1 + y_1^2} - 1\right). \tag{17}$$

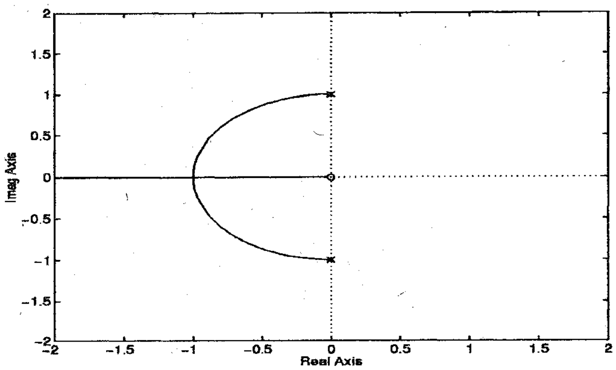

Fig. 4. Root locus for $s^2 + bs + a$ as b varies from 0–∞.

Note that we have substituted the quadratic term by a term with linear growth at infinity. Nevertheless, W_b is positive definite, radially unbounded and $\dot{W}_b = -(\delta - k_1)y_2 \arctan y_2 \le 0$. Again, by LaSalle's invariance principle, the origin is globally asymptotically stable.

(P2) with Prespecified Equilibrium Structure: We may want to preserve the existing equilibria or to create a set of new ones (for example, to prevent unwinding). As an illustration we use the feedback to create the set of equilibria at $(0, 0, 2k\pi, 0)$ with $k = 0, \pm 1, \pm 2, \cdots$. These equilibria will necessarily be alternatively stable and unstable. This objective is achieved by the bounded control

$$u_e = -k_1 \sin(0.5y_1) - k_2 \arctan y_2. \tag{18}$$

The Lyapunov function is

$$W_e = W - \frac{k_1}{2} y_1^2 + 4k_1 \sin^2(0.25y_1) \tag{19}$$

and the invariant set is given by $(0, 0, 2k\pi, 0)$. It can be shown that the states converge to one of these equilibria which represent the same point in physical space.

(P3) Controller—Transient Performance: It has already been pointed out that the (P2) controller cannot decrease the settling time of x_1 beyond a limit. To explain the reasons for this limitation and to motivate our approach to overcome it, we consider the following simple linear system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{20}$$

This system is passive from the input u to the output y with respect to the storage function $V = (a/2)x^2 + \frac{1}{2}y^2$. Now the simplest feedback law which achieves global asymptotic stability is $u = -by$ which corresponds to the (P2) control for the system (9).

The root locus for the characteristic equation of the closed-loop system, as b varies from 0 to ∞ and $a = 1$, is given in Fig. 4.

It shows that the settling time for x cannot be decreased arbitrarily.

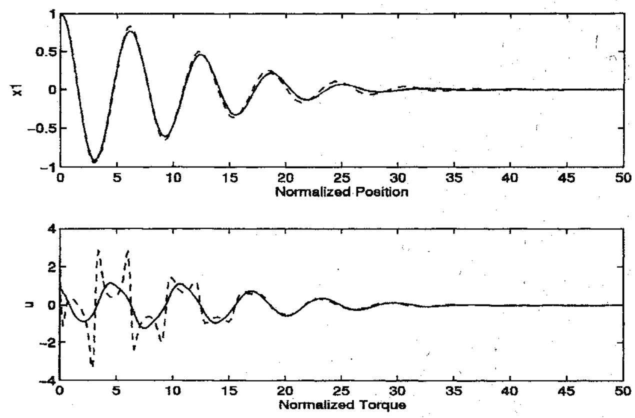

Fig. 5. Response with (P3) and (C2) controllers.

To assign a smaller settling time we have to use the x variable in the feedback law. In the framework of passivating designs this can be accomplished by modifying the storage function to increase the penalty on x. Thus with the storage function $V = (a + c/2)x^2 + \frac{1}{2}y^2$, the resulting control law $u = -cx - by$ achieves arbitrarily small settling time of x by increasing b and c.

Motivated by the above example we introduce a design parameter k_0 to increase the penalty on the z-variables in the Lyapunov function (14)

$$\begin{aligned} W_a = &\frac{k_0 + 1}{2}[(z_1 - \varepsilon \sin y_1)^2 + z_2^2] + \frac{k_1}{2} y_1^2 \\ &+ \tfrac{1}{2} y_2^2 (1 - \varepsilon^2 \cos^2 y_1). \end{aligned} \tag{21}$$

The function W_a can be made a storage function by the passivating feedback transformation $u = -k_0 \varepsilon \cos y_1(-z_1 + \varepsilon \sin y_1) - k_1 y_1 + u_1$. Therefore, our (P3) controller achieving $\dot{W}_a = -k_2 y_2^2 \le 0$ is given by

$$u = -k_0 \varepsilon \cos y_1(-z_1 + \varepsilon \sin y_1) - k_1 y_1 - k_2 y_2 \tag{P3}$$

and LaSalle's invariance principle again guarantees that the origin is globally asymptotically stable. Note that when $k_0 = 0$ this control law reduces to (P2).

This design modification significantly improves the settling time, compared with the (P2) controller. By increasing the parameter k_0 we can match the performance of the cascade controllers, but with smaller control effort as illustrated in Fig. 5. The solid curves represent the (P3) controller with $(k_0, k_1, k_2) = (10, 1, 0.5)$ and the dashed curves represent the (C2) controller with $(p_0, p_1, p_2, c_0, c_1, c_2) = (2, 0.2, 1, 2.3, 0.6, 0.6)$.

Remark 4.1: Note that the controllers (P2) and (P3) guarantee global asymptotic stability of the closed-loop system for any positive values of the gains k_0, k_1, k_2. Therefore, if we multiply the control u by any positive constant, the resulting control law still achieves global asymptotic stability; in other words, these controllers have infinite gain margin.

From (2) we have that $z_1 - \varepsilon \sin y_1 = x_1$. Recall that x_1 is the normalized displacement of the platform. If we denote by q_1 the actual (measured) displacement and by σ the normalizing factor, we can rewrite (P3) as

$$u = k_0 \varepsilon \sigma q_1 \cos \theta - k_1 \theta - k_2 \dot{\theta}.$$

Thus, even when ε and σ are not known (i.e., when the mass parameters of the system are not known), (P3) achieves global asymptotic stability.

V. Conclusion

In this paper we have considered the problem of asymptotic stabilization of the TORA system. Based on two different paradigms we have designed several controllers which achieve global asymptotic stability. Performance of these controllers was compared by simulations. The summary of our findings is given below:

- For the TORA system the (C2) controller and the much simpler (C1) controller give almost the same performance because the coupling between subsystems is weak.
- The (P2) controller uses only θ and $\dot{\theta}$ for feedback and requires the smallest control effort. Its drawback is that the settling time cannot be decreased beyond a certain limit.
- The (P3) controller allows a faster and better damped response at the expense of increased control effort and the addition of a sensor for platform displacement. Still, the control effort is smaller than with the comparable cascade controllers.
- The design of passivating controllers is not systematic because it relies on the knowledge of a storage function. On the other hand, the controller design for cascade systems is systematic.

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