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Nonlinear observers: a circle criterion design and robustness analysis☆

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Abstract

Globally convergent observers are designed for a class of systems with monotonic nonlinearities. The approach is to represent the observer error system as the feedback interconnection of a linear system and a time-varying multivariable sector nonlinearity. Using LMI software, observer gain matrices are computed to satisfy the circle criterion and, hence, to drive the observer error to zero. In output-feedback design, the observer is combined with control laws that ensure input-to-state stability with respect to the observer error. Robustness to unmodeled dynamics is achieved with a small-gain assignment design, as illustrated on a jet engine compressor example.

Keywords: Nonlinear observer design; Output-feedback control; Axial compressors; Robustness; Unmodeled dynamics

1. Introduction

Progress in nonlinear observers has been slower than in other areas of nonlinear control theory. Early results were obtained under a global Lipschitz restriction which excludes common nonlinearities such as x^3, $\exp(x)$, etc. The results of Thau (1973), Kou, Elliott, and Tarn (1975), Banks (1981) and their recent extensions by Raghavan and Hedrick (1994), and Rajamani (1998) make use of quadratic Lyapunov functions for the observer error system. A more advanced framework was introduced by Tsinias (1989, 1993), who pursued an observer analog of *control Lyapunov functions* and used them for output feedback design.

Another common restriction is to require that the nonlinearities appear as functions of the measured output. Systems which admit such a representation have been characterized by Krener and Isidori (1983), Bestle and Zeitz (1983), and other authors. The observer design for these systems is linear because the nonlinearity is canceled by an output injection term.

A design which dominates state-dependent nonlinearities by high-gain linear terms has been developed by Esfandiari and Khalil (1992). To avoid the destabilizing effect of the *peaking phenomenon* analyzed by Sussmann and Kokotović (1991), this design employs judicious scaling of the observer gain matrix, and saturation of the peaking observer signals before they are fed to the controller. To achieve global convergence of high-gain observers, Gauthier, Hammouri, and Othman (1992) resorted to a global Lipschitz restriction.

The design presented in this paper removes the global Lipschitz restriction and avoids high-gain. Instead, it makes two restrictions which allow the observer error system to satisfy the well-known *multivariable circle criterion.* First, a linear matrix inequality (LMI) is to be feasible, which implies a *strict positive real* (SPR) property for the linear part of the observer error system. The second restriction is that the nonlinearities be nondecreasing functions of linear combinations of unmeasured states. This restriction ensures that the vector time-varying nonlinearity in the observer error system satisfies the sector condition of the circle criterion. A preliminary idea of this global observer design was presented in Arcak and Kokotović (1999) for systems in *strict-feedback* form. In the current paper we develop a complete design

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procedure for more general systems and provide a robustness analysis for observer-based feedback design.

In the proposed design, presented in Section 2, observer matrices satisfying the circle criterion are evaluated by efficient LMI computations. The robustness property against inexact modeling of nonlinearities is established, and the feasibility of a reduced-order variant of the observer is shown to be equivalent to that of the full-order observer. In Section 3, the new observer is incorporated in output-feedback design with control laws that ensure *input-to-state stability* (ISS) with respect to the observer error. For unmodeled dynamics such as those studied by Praly and Jiang (1993), a small-gain assignment design achieves robust observer-based stabilization. As illustrated by a jet engine compressor example in Section 4, this design broadens the applicability of the new observer to systems which violate feasibility of the exact observer design.

2. Nonlinear observers: circle criterion design

For our observer design, we consider the plant

$$\dot{x} = Ax + G\gamma(Hx) + \varrho(y,u),$$
$$y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^m$ is the control input, the pair (A, C) is detectable, and, $\gamma(\cdot)$ and $\varrho(\cdot,\cdot)$ are locally Lipschitz. The state-dependent nonlinearity $\gamma(Hx)$ is an r-dimensional vector where each entry is a function of a linear combination of the states

$$\gamma_i = \gamma_i\left(\sum_{j=1}^{n} H_{ij}x_j\right), \quad i = 1,\dots,r. \tag{2}$$

Our main restriction is that each $\gamma_i(\cdot)$ be nondecreasing, that is, for all $a, b \in \mathbb{R}$, it satisfies

$$(a-b)[\gamma_i(a) - \gamma_i(b)] \geqslant 0. \tag{3}$$

If $\gamma_i(\cdot)$ is continuously differentiable, then $\mathrm{d}\gamma_i(v)/\mathrm{d}v \geqslant 0$ for all $v \in \mathbb{R}$. If, instead, $\gamma_i(\cdot)$ satisfies $\mathrm{d}\gamma_i(v)/\mathrm{d}v \geqslant g_i$, $g_i \neq 0$, we can still represent the system as in (1)–(3) by defining a new function $\tilde{\gamma}_i(v) := \gamma_i(v) - g_i v$ which satisfies $\mathrm{d}\tilde{\gamma}_i(v)/\mathrm{d}v \geqslant 0$. When the nonlinearity $\gamma(Hx)$ also depends on y and u, the nondecreasing property (3) is to hold for each $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$.

For the plant (1), we construct an observer in the form

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + G\gamma(H\hat{x} + K(C\hat{x} - y)) + \varrho(y,u). \tag{4}$$

Our task is to determine the observer matrices $K \in \mathbb{R}^{r \times p}$ and $L \in \mathbb{R}^{n \times p}$. At this point, we assume that the solution $x(t)$ of (1) does not escape to infinity in finite time. In Section 3, this assumption will be satisfied by our output feedback control design.

From (1) and (4), the dynamics of the observer error $e = x - \hat{x}$ are governed by

$$\dot{e} = (A + LC)e + G[\gamma(v) - \gamma(w)], \tag{5}$$

where

$$v := Hx, \quad w := H\hat{x} + K(C\hat{x} - y). \tag{6}$$

We begin the observer design by representing the observer error system (5) as the feedback interconnection of a linear system and a multivariable sector nonlinearity. To this end, we view $\gamma(v) - \gamma(w)$ as a function of v and $z := v - w = (H + KC)e$; that is, a time-varying nonlinearity in z:

$$\varphi(t,z) := \gamma(v) - \gamma(w), \tag{7}$$

where the time dependence is due to $v(t)$. Substituting (7), we rewrite the observer error system (5) as

$$\dot{e} = (A + LC)e + G\varphi(t,z),$$
$$z = (H + KC)e \tag{8}$$

and note from (3) that each component of $\varphi(t,z)$ satisfies

$$z_i\varphi_i(t,z_i) \geqslant 0, \quad \forall z_i \in \mathbb{R}. \tag{9}$$

Thanks to this sector property of each $\varphi_i(t,z_i)$, the product $\varphi(t,z)^{\mathrm{T}}\Lambda z$ is nonnegative for any diagonal $\Lambda > 0$. This means that the feedback path in the observer error system depicted in Fig. 1 is a multivariable sector nonlinearity. Thus, from the circle criterion, asymptotic stability is guaranteed if the linear system with input ϑ and output Λz is SPR, that is, if a matrix $P = P^{\mathrm{T}} > 0$, a constant $\nu > 0$, and a diagonal matrix $\Lambda > 0$ exist such that

$$\begin{bmatrix} (A + LC)^{\mathrm{T}}P + P(A + LC) + \nu I & PG + (H + KC)^{\mathrm{T}}\Lambda \\ G^{\mathrm{T}}P + \Lambda(H + KC) & 0 \end{bmatrix} \leqslant 0. \tag{10}$$

Thus, the observer design for system (1) consists in finding observer matrices K and L to satisfy (10) with some

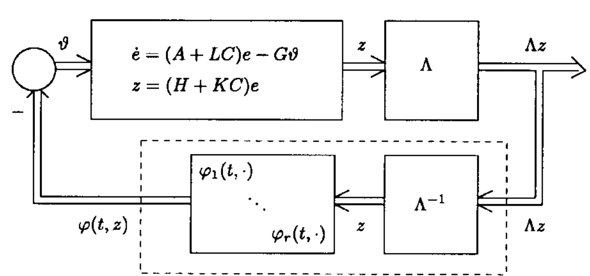

Fig. 1. Observer error system.

$P = P^{\mathrm{T}} > 0$, $\Lambda > 0$, and $\nu > 0$. Efficient numerical tools are available for this task because (10) is a LMI in P, PL, Λ, ΛK and ν. It is important to note that the feasibility of (10) is not known *a priori*, and is to be determined numerically. An analytical test of feasibility has been derived in Arcak (2000).

Robustness issues are critical in every observer design. To analyze the robustness of the above observer against inexact modeling of nonlinearities, we suppose that instead of (1), the plant is

$$\dot{x} = Ax + G[\gamma(Hx) + \Delta(Hx)\mu(t)] + \varrho(y,u), \tag{11}$$

where $\mu(t)$ is an unknown bounded disturbance. Then, using the observer (4), we get the observer error system

$$\dot{e} = (A + LC)e + G[\gamma(v) - \gamma(w) + \Delta(v)\mu(t)], \tag{12}$$

where v and w are as in (6).

We now characterize uncertain nonlinearities $\Delta(\cdot)$ for which the observer (4) guarantees an ISS property from the disturbance $\mu(t)$ to the observer error $e(t)$.

Theorem 1. *Consider the plant* (11) *and the observer* (4). *Suppose $x(t)$ exists for all $t \geqslant 0$, and that the LMI* (10) *holds with a matrix $P = P^{\mathrm{T}} > 0$, a constant $\nu > 0$, and a diagonal matrix $\Lambda > 0$. If, for each $i = 1, \ldots, r$, there exists a class-$\mathscr{K}$ function $\sigma_i(\cdot)$ such that*

$$(a - b)[\gamma_i(a) - \gamma_i(b) + \Delta_i(a)\mu] \geqslant -\sigma_i(|\mu|), \quad \forall a, b, \mu \in \mathbb{R}, \tag{13}$$

then the observer error $e(t)$ satisfies, for all $t \geqslant 0$,

$$|e(t)| \leqslant \kappa |e(0)| \exp(-\beta t) + \rho\left(\sup_{0 \leqslant \tau \leqslant t} |\mu(\tau)|\right), \tag{14}$$

where $\kappa = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$, $\beta = \nu/2\lambda_{\max}(P)$, *and the ISS-gain from $\mu(t)$ to $e(t)$ is*

$$\rho(\cdot) = \kappa \sqrt{\frac{2}{\nu} \sum_{i=1}^{r} \lambda_i \sigma_i(\cdot)}. \tag{15}$$

Proof. We use $V = e^{\mathrm{T}} P e$ as an ISS-Lyapunov function, and evaluate its derivative for (12):

$$\dot{V} \leqslant -\nu |e|^2 - 2 \sum_{i=1}^{r} \lambda_i (v_i - w_i) \times [\gamma_i(v_i) - \gamma_i(w_i) + \Delta_i(v_i)\mu]. \tag{16}$$

Substituting (13), we obtain

$$\dot{V} \leqslant -\nu |e|^2 + 2 \sum_{i=1}^{r} \lambda_i \sigma_i(|\mu|) \leqslant -2\beta V + 2 \sum_{i=1}^{r} \lambda_i \sigma_i(|\mu|) \tag{17}$$

from which it follows that

$$|e(t)| \leqslant \kappa |e(0)| \exp(-\beta t) + \sqrt{\frac{1}{\beta \lambda_{\min}(P)} \left(\sum_{i=1}^{r} \lambda_i \sup_{0 \leqslant \tau \leqslant t} \sigma_i(|\mu(\tau)|) \right)}, \tag{18}$$

so that (14) and (15) result from $1/\beta\lambda_{\min}(P) = 2\kappa^2/\nu$. □

The ISS property established by Theorem 1 shows that $e(t)$ decreases with the decrease in the magnitude of the disturbance $\mu(t)$. As $\mu(t)$ vanishes, we recover exponential convergence of the observer error to zero.

The dependence of uncertain nonlinearities $\Delta(\cdot)$ on $\gamma(\cdot)$ is characterized by (13). For example, if $\gamma(\cdot)$ is cubic, then $\Delta(\cdot)$ is allowed to be linear. In this case, (13) is satisfied because

$$(a - b)[a^3 - b^3 + a\mu] \geqslant -\tfrac{1}{3}\mu^2 \tag{19}$$

holds for all $a, b, \mu \in \mathbb{R}$ due to the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

In applications it may be more convenient to employ a reduced-order observer, which generates estimates only for the unmeasured states. The design of such an observer starts with a preliminary change of coordinates such that the output y consists of the first p entries of the state vector $x = [\, y^{\mathrm{T}} \;\; x_0^{\mathrm{T}}]^{\mathrm{T}}$. In the new coordinates, the system (1) is

$$\begin{aligned} \dot{y} &= A_1 x_0 + G_1 \gamma(H_1 y + H_2 x_0) + \varrho_1(y,u), \\ \dot{x}_0 &= A_2 x_0 + G_2 \gamma(H_1 y + H_2 x_0) + \varrho_2(y,u), \end{aligned} \tag{20}$$

where the linear terms in y are incorporated in $\varrho_1(y,u)$ and $\varrho_2(y,u)$. An estimate of x_0 will be obtained via $\chi := x_0 + Ny$, where $N \in \mathbb{R}^{(n-p)\times p}$ is to be designed. From (20), the derivative of χ is

$$\dot{\chi} = (A_2 + NA_1)\chi + (G_2 + NG_1)\gamma(H_2 \chi + (H_1 - H_2 N)y) + \bar{\varrho}(y,u), \tag{21}$$

where $\bar{\varrho}(y,u) := N\varrho_1(y,u) + \varrho_2(y,u) - (A_2 + NA_1)Ny$. In this χ-subsystem, the output injection matrix N has altered the A_2 and G_2 matrices of the x_0-subsystem (20).

To obtain the estimate

$$\hat{x}_0 = \hat{\chi} - Ny, \tag{22}$$

we employ the observer

$$\dot{\hat{\chi}} = (A_2 + NA_1)\hat{\chi} + (G_2 + NG_1)\gamma(H_2 \hat{\chi} + (H_1 - H_2 N)y) + \bar{\varrho}(y,u). \tag{23}$$

From (21) and (23), the dynamics of $e_0 := x_0 - \hat{x}_0 = \chi - \hat{\chi}$ are governed by

$$\dot{e}_0 = (A_2 + NA_1)e_0 + (G_2 + NG_1)[\gamma(v_0) - \gamma(w_0)], \tag{24}$$

where $v_0 := H_2 \chi + (H_1 - H_2 N)y$ and $w_0 := H_2 \hat{\chi} + (H_1 - H_2 N)y$. We let $z := v_0 - w_0 = H_2 e_0$, and denote

$\varphi(t,z) = \gamma(v_0) - \gamma(w_0)$. Then, the nondecreasing property (3) implies that $z_i\varphi_i(t,z_i) \geqslant 0$ for all $i = 1,\dots,r$, and derivations similar to those in Section 2 yield the following LMI in P_0, P_0N, ν and Λ:

$$\begin{bmatrix} (A_2+NA_1)^{\mathrm{T}}P_0 + P_0(A_2+NA_1) + \nu I & P_0(G_2+NG_1) + H_2^{\mathrm{T}}\Lambda \\ (G_2+NG_1)^{\mathrm{T}}P_0 + \Lambda H_2 & 0 \end{bmatrix} \leqslant 0. \qquad (25)$$

If this LMI is satisfied with a matrix $P_0 = P_0^{\mathrm{T}} > 0$, a constant $\nu \geqslant 0$, and a diagonal matrix $\Lambda > 0$, then it is not difficult to show that, with appropriate modifications, Theorem 1 holds for the observer error $e_0(t)$. Furthermore, as we prove in Appendix A.1, the reduced-order observer exists whenever the full-order observer exists:

Theorem 2. *Let the constant* $\nu \geqslant 0$ *and the diagonal matrix* $\Lambda > 0$ *be given. Then, the following two statements are equivalent*:

1. *There exist matrices* $P = P^{\mathrm{T}} > 0$, *K and L satisfying the full-order observer LMI* (10).
2. *There exist matrices* $P_0 = P_0^{\mathrm{T}} > 0$ *and N satisfying the reduced-order observer LMI* (25).

To emphasize the importance of the concept of injecting an output signal in the observer nonlinearity ($K \neq 0$), we provide an example in which the observer cannot be designed with $K = 0$:

Example 1. With $y = x_1$, and the nondecreasing nonlinearity $\gamma(x_2) = x_2^5$, the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2, \\ \dot{x}_2 &= x_2 - x_2^5 + u \end{aligned} \qquad (26)$$

is of the form (1) with $C = [1\ 0]$, $G = [0\ \ -1]^{\mathrm{T}}$, $H = [0\ 1]$. Because the off-diagonal term in the LMI (10) must be zero, the restriction $K = 0$ implies that P be diagonal. However, because

$$A + LC = \begin{bmatrix} 1 + l_1 & 1 \\ l_2 & 1 \end{bmatrix},$$

it is not difficult to show that $(A+LC)^{\mathrm{T}}P + P(A+LC)$ cannot be negative definite with a positive definite diagonal P.

With $K \neq 0$ the design is feasible and a solution of the LMI (10) is $L = [-2\ \ -3]^{\mathrm{T}}$, $K = -1.5$. The resulting observer is

$$\begin{aligned} \dot{\hat{x}}_1 &= -2(\hat{x}_1 - y) + \hat{x}_2 + y, \\ \dot{\hat{x}}_2 &= -3(\hat{x}_1 - y) + \hat{x}_2 - (\hat{x}_2 - 1.5(\hat{x}_1 - y))^5 + u. \end{aligned} \qquad (27)$$

Likewise, a reduced-order observer for $\hat{x}_2 = \hat{\chi} + 1.5y$ is

$$\dot{\hat{\chi}} = -0.5\hat{\chi} - (\hat{\chi} + 1.5y)^5 - 2.25y + u. \qquad (28)$$

3. Output-feedback control

The design of a nonlinear output-feedback control law using the certainty equivalence implementation of a full state-feedback law is, in general, impossible as we illustrate on the system (26). This system is stabilizable with full state feedback

$$u = k_1x_1 + k_2x_2 + x_2^5. \qquad (29)$$

When only $y = x_1$ is measured, the certainty equivalence control law (29) with the unmeasured state x_2 replaced with its estimate $\hat{x}_2$ results in the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2, \\ \dot{x}_2 &= k_1x_1 + (k_2+1)x_2 + [(x_2 - e_2(t))^5 - x_2^5] \\ &\quad - k_2e_2(t), \end{aligned} \qquad (30)$$

where $e_2(t) = x_2(t) - \hat{x}_2(t)$ is exponentially decaying. As we prove in Appendix A.2, this system exhibits finite escape time for any choice of k_1 and k_2.

Rather than attempting a certainty equivalence design, we view the state observer error $e = x - \hat{x}$ as a disturbance acting on the system (1), and design a control law $u = \alpha(y,\hat{x}) = \alpha(y, x - e)$ to render the system ISS with respect to the observer error $e(t)$. Such ISS control laws have already been designed for classes of nonlinear systems in Freeman and Kokotović (1996), Krstić, Kanellakopoulos, and Kokotović (1995), and Marino and Tomei (1995). Here, we pursue a more general output-feedback design, where we achieve robustness against dynamic modeling errors. We consider the plant

$$\dot{x} = Ax + G[\gamma(Hx) + \Delta(Hx)\mu] + \varrho(y,u), \quad y = Cx, \qquad (31)$$

$$\begin{aligned} \dot{\xi} &= q(\xi, h(x)), \\ \mu &= p(\xi, h(x)), \end{aligned} \qquad (32)$$

where $\gamma(\cdot)$ and $\Delta(\cdot)$ are as in (13), and the ξ-subsystem (32) represents unmodeled dynamics. This formulation extends the applicability of our observer because, if there is a subsystem to which the observer is not applicable, it can be treated as unmodeled dynamics.

The unmodeled dynamics (32) are assumed to possess the following input-to-output stability (IOS), and ISS properties:

$$|\mu(t)| \leqslant \max\left\{\beta_\mu(|\xi(0)|,t), \rho_{\mu h}\left(\sup_{0\leqslant\tau\leqslant t}|h(x(\tau))|\right)\right\}, \qquad (33)$$

$$|\xi(t)| \leqslant \max\left\{\beta_\xi(|\xi(0)|,t), \rho_{\xi h}\left(\sup_{0\leqslant\tau\leqslant t}|h(x(\tau))|\right)\right\}, \qquad (34)$$

where $\beta_\mu(\cdot,\cdot)$, $\beta_\xi(\cdot,\cdot)$ are class-$\mathscr{KL}$ functions, and $\rho_{\mu h}(\cdot)$, $\rho_{\xi h}(\cdot)$ are class-$\mathscr{K}$ functions.

The ISS property of the observer error (14) is now rewritten in the 'max' form of Teel (1996), which is

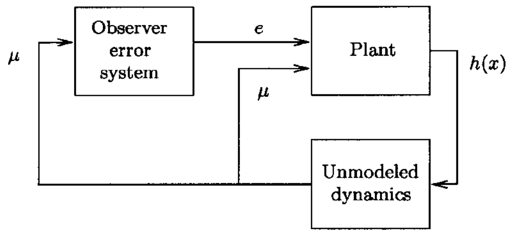

Fig. 2. Closed-loop system with observer feedback.

suitable for the small-gain analysis we will employ:

$$|e(t)| \leqslant \max\left\{\beta_e(|e(0)|,t), \rho_{e\mu}\left(\sup_{0\leqslant\tau\leqslant t}|\mu(\tau)|\right)\right\}. \tag{35}$$

To prepare for a small-gain design of $u = \alpha(y,\hat{x})$, we represent the closed-loop system with the block-diagram in Fig. 2. The gains of the unmodeled dynamics block and the observer error block are $\rho_{\mu h}(\cdot)$ and $\rho_{e\mu}(\cdot)$, respectively. Let $\rho_{he}(\cdot)$ and $\rho_{h\mu}(\cdot)$ be the plant gains from e and μ to $h(x)$, respectively. The task for the control law $u = \alpha(y,\hat{x})$ is to render $\rho_{he}(\cdot)$ and $\rho_{h\mu}(\cdot)$ small enough for the inner loop gain $\rho_{h\mu}\circ\rho_{\mu h}(\cdot)$, and the outer loop gain $\rho_{he}\circ\rho_{e\mu}\circ\rho_{\mu h}(\cdot)$ to satisfy, for all $s > 0$,

$$\rho_{h\mu}\circ\rho_{\mu h}(s) < s, \tag{36}$$

$$\rho_{he}\circ\rho_{e\mu}\circ\rho_{\mu h}(s) < s. \tag{37}$$

Then, global asymptotic stability (GAS) of the closed-loop system will be guaranteed as in the *nonlinear small-gain theorem* of Jiang, Teel, and Praly (1994), and Teel (1996).

Theorem 3. *Consider the system* (31) *and* (32), *in which* $\gamma(\cdot)$ *and* $\Delta(\cdot)$ *satisfy* (13), *and the* ξ*-subsystem satisfies* (33) *and* (34). *Suppose that the observer*

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + G\gamma(H\hat{x} + K(C\hat{x} - y)) + \varrho(y,u) \tag{38}$$

is such that the LMI (10) *holds with a matrix* $P = P^{\mathrm{T}} > 0$, *a constant* $v > 0$, *and a diagonal matrix* $\Lambda > 0$. *If the control law* $u = \alpha(y,\hat{x})$ *guarantees*

$$|h(x(t))| \leqslant \max\left\{\beta_h(|x(0)|,t),\ \rho_{h\mu}\left(\sup_{0\leqslant\tau\leqslant t}|\mu(\tau)|\right),\ \rho_{he}\left(\sup_{0\leqslant\tau\leqslant t}|e(\tau)|\right)\right\} \tag{39}$$

$$|x(t)| \leqslant \max\left\{\beta_x(|x(0)|,t),\ \rho_{x\mu}\left(\sup_{0\leqslant\tau\leqslant t}|\mu(\tau)|\right),\ \rho_{xe}\left(\sup_{0\leqslant\tau\leqslant t}|e(\tau)|\right)\right\}, \tag{40}$$

where $\rho_{h\mu}(\cdot)$ *and* $\rho_{he}(\cdot)$ *satisfy* (36) *and* (37), *respectively, then the origin of the closed-loop system* (31), (32) *and* (38) *is globally asymptotically stable.*

4. Design example

An axial compressor model, which has been the starting point for jet engine control studies, is the following single-mode approximation of a PDE model due to Moore and Greitzer (1986),

$$\dot{\phi} = -\psi + \tfrac{3}{2}\phi + \tfrac{1}{2} - \tfrac{1}{2}(\phi+1)^3 - 3(\phi+1)R, \tag{41}$$

$$\dot{\psi} = \frac{1}{\beta^2}(\phi + 1 - u), \tag{42}$$

$$\dot{R} = \sigma R(-2\phi - \phi^2 - R), \quad R(0) \geqslant 0, \tag{43}$$

where ϕ and ψ are the deviations of the mass flow and the pressure rise from their set points, the control input u is the flow through the throttle, and, σ and β are positive constants. This model captures the main surge instability between the mass flow and the pressure rise. It also incorporates the nonnegative magnitude R of the first stall mode.

A state feedback GAS control law in (Krstić et al., 1995, Section 2.4) was replaced in Krstić, Fontaine, Kokotović, and Paduano (1998) by a design using ϕ and ψ. With a high-gain observer, (Isidori, 1999, Section 12.7), and Maggiore and Passino (2000) obtained a semiglobal result using the measurement of ψ alone. With $y = \psi$, we will now achieve GAS for (41)–(43). While the exact observer cannot be designed because of the nonlinearities ϕR and $\phi^2 R$, the (ϕ,ψ)-subsystem (41) and (42) contains the nondecreasing nonlinearity $(\phi+1)^3$, and is of the form (31) with disturbance $\mu = R$. This suggests that we treat the R-subsystem (43) as unmodeled dynamics and apply the small-gain design of Section 3.

First, we prove that $\mu = R$ satisfies the IOS property (33) with $h(x) = \phi$ as the input. With $V = R^2$ as an ISS-Lyapunov function, $R \geqslant 2.1|\phi|$ implies $\dot{V} \leqslant -0.09\sigma R^3$, because $R(t) \geqslant 0$ for all $t \geqslant 0$. This means that (33) holds with the linear gain

$$\rho_{\mu h}(\cdot) = 2.1(\cdot), \tag{44}$$

and, since $\mu = \xi = R$, the ISS property (34) is also satisfied.

To design the reduced-order observer of Section 2 for the (ϕ,ψ)-subsystem (41) and (42), we let $\chi = \phi + N\psi$, and obtain

$$\dot{\chi} = \left(\frac{3}{2} + \frac{N}{\beta^2}\right)\chi - \frac{1}{2}(\chi - N\psi + 1)^3 - 3(\chi - N\psi + 1)R + \bar{\varrho}(\psi,u), \tag{45}$$

where

$$\bar{\varrho}(\psi,u) := -\left(\frac{3}{2}+\frac{N}{\beta^2}\right)N\psi - \psi + \frac{1}{2} + \frac{N}{\beta^2}(1-u). \quad (46)$$

The resulting observer for $\hat{\phi} = \hat{\chi} - N\psi$ is the scalar equation

$$\dot{\hat{\chi}} = \left(\frac{3}{2}+\frac{N}{\beta^2}\right)\hat{\chi} - \frac{1}{2}(\hat{\chi} - N\psi + 1)^3 + \bar{\varrho}(\psi,u). \quad (47)$$

For its implementation, the LMI (25) is satisfied by selecting N such that

$$k := -\left(\frac{3}{2}+\frac{N}{\beta^2}\right) > 0. \quad (48)$$

To prove the ISS property (35) for the observer error $e_\phi = \phi - \hat{\phi}$, we employ the ISS-Lyapunov function $V_e = e_\phi^2$, and evaluate its derivative for

$$\dot{e}_\phi = -ke_\phi - \tfrac{1}{2}(a^3 - b^3 + 6aR), \quad (49)$$

where $a := \chi - N\psi + 1$ and $b = \hat{\chi} - N\psi + 1$. Employing the inequality (19), and substituting $a - b = e_\phi$, we get

$$\dot{V}_e \leqslant -2ke_\phi^2 + 12R^2, \quad (50)$$

from which $|e_\phi| \geqslant \sqrt{6.1/k}|R|$ implies $\dot{V}_e \leqslant -0.03ke_\phi^2$, and, hence, the ISS property (35) holds with the linear gain

$$\rho_{e\mu}(\cdot) = \sqrt{\frac{6.1}{k}}(\cdot). \quad (51)$$

We are now ready to design a control law as in Theorem 3. Noting that (41) and (42) is in strict feedback form, we apply one step of observer backstepping. For ψ, we design the virtual control law $\alpha_0 = c_1\hat{\phi}$. Denoting

$$\omega := \psi - c_1\hat{\phi} = \psi - c_1\phi + c_1 e_\phi, \quad (52)$$

we rewrite (41) as

$$\dot{\phi} = -c_1\phi - \tfrac{3}{2}\phi^2 - \tfrac{1}{2}\phi^3 - 3\phi R - \omega - 3R + c_1 e_\phi. \quad (53)$$

The substitution of (47) in (52) yields $\omega = (1 + Nc_1)\psi - c_1\hat{\chi}$, and, from (42) and (47),

$$\dot{\omega} = \frac{1+Nc_1}{\beta^2}\phi + \frac{1}{\beta^2}(1-u) + \Gamma(\hat{\phi},\psi), \quad (54)$$

where $\Gamma(\hat{\phi},\psi) := c_1\psi + c_1k\hat{\phi} + c_1/2(\hat{\phi}+1)^3 - c_1/2$. Then, the control law

$$u = 1 + (1+Nc_1)\hat{\phi} + \beta^2(c_2\omega + \Gamma(\hat{\phi},\psi)) \quad (55)$$

is implementable using the signals ψ and $\hat{\phi}$, and results in

$$\dot{\omega} = -c_2\omega + \frac{1+Nc_1}{\beta^2}e_\phi. \quad (56)$$

The remaining task is to select the design parameters c_1 and c_2 such that (39) and (40) are satisfied. For the ISS-Lyapunov function $W(\phi,\omega) := \frac{1}{2}\phi^2 + \frac{1}{2}\omega^2$, the inequalities $-\frac{3}{2}\phi^3 \leqslant \frac{9}{8}\phi^2 + \frac{1}{2}\phi^4$, $-\phi\omega \leqslant \frac{1}{2}\phi^2 + \frac{1}{2}\omega^2$, $-3\phi R \leqslant \frac{9}{4}\phi^2 + R^2$, $-3\phi^2 R \leqslant 0$ (because $R(t) \geqslant 0$), $c_1\phi e_\phi \leqslant (c_1/2)\phi^2 + (c_1/2)e_\phi^2$, and $(1 + Nc_1/\beta^2)\omega e_\phi \leqslant ((1+Nc_1)^2/2\beta^4 c_1)\omega^2 + (c_1/2)e_\phi^2$, yield

$$\dot{W} \leqslant -\left(\frac{c_1}{2} - \frac{31}{8}\right)\phi^2 - \left(c_2 - \frac{1}{2} - \frac{(1+Nc_1)^2}{2\beta^4 c_1}\right)\omega^2 + R^2 + c_1 e_\phi^2. \quad (57)$$

We let $c > 0$, and select c_1 and c_2 to satisfy

$$\left(\frac{c_1}{2} - \frac{31}{8}\right) > c, \qquad \left(c_2 - \frac{1}{2} - \frac{(1+Nc_1)^2}{2\beta^4 c_1}\right) > c, \quad (58)$$

so that

$$\dot{W} \leqslant -c(\phi^2 + \omega^2) + R^2 + (2c + \tfrac{31}{4})e_\phi^2, \quad (59)$$

from which (40) follows for $x = (\phi,\psi)$. For $h(x) = \phi$ and $\mu = R$, we now compute the gains $\rho_{h\mu}(\cdot)$ and $\rho_{he}(\cdot)$ in (39). Using the fact that for each constant $\theta > 0$, $a + b \leqslant \max\{(1+\theta^{-1})a, (1+\theta)b\}$ for all $a, b \geqslant 0$, we obtain

$$\begin{aligned}\dot{W} &\leqslant -c(\phi^2+\omega^2) + R^2 + (2c + \tfrac{31}{4})e_\phi^2 \\ &\leqslant -2cW + \max\{(1+\theta^{-1})R^2, (1+\theta)(2c + \tfrac{31}{4})e_\phi^2\},\end{aligned} \quad (60)$$

from which it follows that

$$W \geqslant \max\left\{\frac{(1+\theta^{-1})}{1.9c}R^2, \frac{(1+\theta)}{1.9c}\left(2c + \frac{31}{4}\right)e_\phi^2\right\} \Rightarrow \dot{W} \leqslant -0.1cW. \quad (61)$$

Then, (39) follows because $|\phi| \leqslant \sqrt{2W}$, and the gains are

$$\rho_{h\mu}(\cdot) = \sqrt{\frac{(1+\theta^{-1})}{0.95c}}(\cdot),$$

$$\rho_{he}(\cdot) = \sqrt{(1+\theta)\left(\frac{2}{0.95} + \frac{31}{3.8c}\right)}(\cdot). \quad (62)$$

Using (44), (51) and (62), the inner and outer loop small-gain conditions, (36) and (37) are, respectively,

$$2.1\sqrt{\frac{(1+\theta^{-1})}{0.95c}} < 1, \quad (63)$$

$$2.1\sqrt{\frac{6.1}{k}}\sqrt{(1+\theta)\left(\frac{2}{0.95} + \frac{31}{3.8c}\right)} < 1. \quad (64)$$

Selecting $c > 0$ and $k > 0$ sufficiently large ensures that (63) and (64) hold. Additional freedom for the selection of

c and k is obtained from $\theta > 0$, which allocates the inner and outer loop gains.

5. Conclusion

Output-feedback control of nonlinear systems continues to be a challenging research area in which every new effort attempts to remove previous restrictions and improve robustness. The advance made in this paper is to allow the presence of nondecreasing nonlinearities to depend on linear combinations of unmeasured states. Global convergence of the nonlinear observer is achieved without the usual global Lipschitz restriction or high-gain feedback. Instead, admissible nonlinearities and system structures are determined via verifiable feasibility conditions, while the design is carried out with computationally efficient LMI software. The ubiquitous question of observer and closed-loop system robustness is also answered for a significant class of modeling errors. It is hoped that these results will reinvigorate the search for new structure-specific nonlinear observer designs.

Appendix A

A.1. Proof of Theorem 2

($1 \Rightarrow 2$) Suppose the full-order observer LMI (10) is feasible for the system (20). Partitioning P and L as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{A.1}$$

and substituting

$$A = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = [H_1 \quad H_2],$$
$$\text{and} \quad C = [I \quad 0] \tag{A.2}$$

in (10), we obtain

$$(A + LC)^{\mathrm{T}}P + P(A + LC) + \nu I$$
$$= \begin{bmatrix} \star & \star \\ \star & P_3(A_2 + P_3^{-1}P_2^{\mathrm{T}}A_1) + (A_2 + P_3^{-1}P_2^{\mathrm{T}}A_1)^{\mathrm{T}}P_3 + \nu I \end{bmatrix}$$
$$\leqslant 0, \tag{A.3}$$

$$PG + (H + KC)^{\mathrm{T}}\Lambda$$
$$= \begin{bmatrix} \star \\ P_3(G_2 + P_3^{-1}P_2^{\mathrm{T}}G_1) + H_2^{\mathrm{T}}\Lambda \end{bmatrix} = 0. \tag{A.4}$$

Defining $P_0 := P_3$, and $N := P_3^{-1}P_2^{\mathrm{T}}$, we note that (A.3) and (A.4) imply

$$P_0(A_2 + NA_1) + (A_2 + NA_1)^{\mathrm{T}}P_0 + \nu I \leqslant 0, \tag{A.5}$$

$$P_0(G_2 + NG_1) + H_2^{\mathrm{T}}\Lambda = 0, \tag{A.6}$$

which is the reduced-order observer LMI (25).

($2 \Rightarrow 1$) Suppose (69) and (70) hold for the system (20). To prove the existence of $P = P^{\mathrm{T}} > 0$, K and L satisfying (10), we rewrite system (20) in y and $\chi = x_0 + Ny$ coordinates, so that

$$A + LC = \begin{bmatrix} \tilde{L}_1 & A_1 \\ \tilde{L}_2 & A_2 + NA_1 \end{bmatrix},$$

$$G = \begin{bmatrix} G_1 \\ G_2 + NG_1 \end{bmatrix}, \tag{A.7}$$

$$H + KC = [H_1 - H_2N + K \quad H_2], \tag{A.8}$$

where $\tilde{L}_1 := L_1 - A_1N$ and $\tilde{L}_2 = L_2 - (A_2 + NA_1)N$. We let

$$P = \begin{bmatrix} I & 0 \\ 0 & P_0 \end{bmatrix} \tag{A.9}$$

and obtain

$$(A + LC)^{\mathrm{T}}P + P(A + LC)$$
$$= \begin{bmatrix} \tilde{L}_1 + \tilde{L}_1^{\mathrm{T}} & A_1 + \tilde{L}_2^{\mathrm{T}}P_0 \\ P_0\tilde{L}_2 + A_1^{\mathrm{T}} & P_0(A_2 + NA_1) + (A_2 + NA_1)^{\mathrm{T}}P_0 \end{bmatrix},$$

$$PG + (H + KC)^{\mathrm{T}}\Lambda$$
$$= \begin{bmatrix} G_1 + (H_1 - H_2N + K)^{\mathrm{T}}\Lambda \\ P_0(G_2 + NG_1) + H_2^{\mathrm{T}}\Lambda \end{bmatrix}. \tag{A.10}$$

In view of (A.5) and (A.6), the choice

$$\tilde{L}_1 = -\frac{\nu}{2}I, \quad \tilde{L}_2 = -P_0^{-1}A_1^{\mathrm{T}},$$
$$\mathrm{K} = -H_1 + H_2N - \Lambda^{-1}G_1^{\mathrm{T}} \tag{A.11}$$

results in the full-order observer LMI (10).

A.2. Proof of finite escape time in system (30)

To prove that system (30) exhibits finite escape time for any choice of k_1 and k_2, we use a variant of Lemma 3 from Mazenc, Praly, and Dayawansa (1994):

Lemma 1. *Consider the system*

$$\dot{\zeta}_1 = \zeta_2,$$
$$\dot{\zeta}_2 = f_2(\zeta_1, \zeta_2, t), \tag{A.12}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$, and $f_2 \in C^1$. If there exists a nonempty open set $U \subset \mathbb{R}$, and positive constants a, b, d, s and $n > 2$ such that, for all $t \in [0, s]$, $\zeta_1 \in U$ and $\zeta_2 > d$,

$$b\zeta_2^n < f_2(\zeta_1, \zeta_2, t) < a\zeta_2^n, \tag{A.13}$$

then, for $\zeta_1(0) \in U$ and sufficiently large $\zeta_2(0) > d$, $\zeta_2(t)$ escapes to infinity in finite time.

With the new variables $\zeta_1 := \exp(-t)x_1$ and $\zeta_2 := \exp(-t)x_2$, system (30) is transformed into the form (A.12) with

$$f_2(\zeta_1,\zeta_2,t) = k_1\zeta_1 + k_2\zeta_2 + \exp(-t) \times \{[(x_2 - e_2(t))^5 - x_2^5] - k_2 e_2(t)\}. \quad (A.14)$$

Noting that the leading term of $[(x_2 - e_2(t))^5 - x_2^5]$ is $-5e_2(t)x_2^4$, it is not difficult to show that the hypotheses of Lemma 1 hold with $n = 4$ on any compact time interval $t \in [0,s]$ in which $e_2(t)$ is negative. Thus, with $e_2(0) < 0$, and $x_2(0) > 0$ sufficiently large, the closed-loop system (30) exhibits finite escape time.

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