Output Regulation of Nonlinear Systems

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Abstract—**The purpose of this paper is to solve the problem of controlling a fixed nonlinear plant in order to have its output track (or reject) a family of reference (or disturbance) signals produced by some external generator. It is shown that, under standard assumptions, this problem is solvable if and only if a certain nonlinear partial differential equation is solvable. Once a solution of this equation is available, a feedback law which solves the problem can easily be constructed. The theory developed in the paper incorporates earlier results established for linear systems.**

I. Introduction

One of the most important problems in linear multivariable control theory is that of controlling a fixed plant in order to have its output tracking (or rejecting) reference (or disturbance) signals produced by some external generator (the exosystem). This problem has been treated by several authors, among whom we mention, for instance, Smith and Davison [15], Francis and Wonham [6], Francis [5], and Wonham [17]. In particular, the work [5] of Francis has shown that the solvability of a multivariable linear regulator problem corresponds to the solvability of a system of two linear matrix equations and this is in turn equivalent, as illustrated by Hautus [7], to a certain property of the transmission polynomials of a composite system which incorporates the plant and the exosystem. The work of Francis and Wonham [6] has shown that, in the case of error feedback, any regulator which solves the problem in question incorporates a model of the dynamical system generating the reference and/or the disturbance signal which must be tracked and/or rejected. This property is commonly known as the *internal model principle*.

More recently, several authors have considered the corresponding problem in a nonlinear setting. The work of Hepburn and Wonham [8] presents a complete extension of the internal model theory to the setting of nonlinear systems defined on differentiable manifolds. The work of Anantharam and Desoer [1] investigates conditions for the existence of regulators for the purpose of tracking constant reference signals. The work of Di Benedetto [4] describes conditions for the existence of regulators in the case of the one-dimensional exosystem. The very recent work of Jie and Rugh [9] presents a complete extension of the conditions established by Francis to the case in which the exosystem generates constant reference signals.

In this paper, we show how the results established by Francis can be extended to the general case in which the exosystem generates time-varying reference signals and/or disturbances (including the very important case of periodic signals), and how the interpretation established by Hautus finds its natural nonlinear extension in terms of the (recently introduced) notion of *zero dynamics*, which is the nonlinear analog of the notion of transmission zeros. More specifically, in the first part of the paper (Sections II–V), we point out that the linear matrix equations introduced by Francis are particular cases of nonlinear equations which express the existence of a (locally defined) manifold which can be rendered invariant by feedback and annihilates the tracking error (the latter being defined as the difference between actual and reference output). We show that, as proved in [5] for linear systems, if the plant is exponentially stabilizable via state feedback (respectively, via output feedback) the solvability of these equations implies the solvability of the regulator problem via state feedback (respectively, via error feedback). In addition, we show that the solvability of these equations is a necessary condition for the existence of a regulator, as long as the reference signals—although assumed to be bounded—do not trivially converge to zero as time tends to infinity. In the second part of the paper (Sections VI–VIII), we show that the existence conditions can be expressed as a special property of the zero dynamics of a composite system which incorporates the plant dynamics, the exosystem dynamics, and whose output is defined as the tracking error. This interpretation lends itself, in some particular cases, to a very expressive test for the existence of nonlinear regulators, as well as to very simple design procedures. In the Appendix, we summarize—for the reader's convenience—some basic facts about center manifold theory that are extensively used throughout the paper.

The results illustrated in this paper owe their inspiration to a series of recent advances in the geometric analysis of the asymptotic properties of nonlinear differential equations and control systems. These include, in particular, the center manifold theory (see Carr [3]), the new methods of analysis and design of control systems based on existence and properties of invariant manifolds developed by Kokotovic [11], [12] (see also Knobloch and Aulbach [13]), and the nonlinear analog of the notion of transmission zeros developed by the authors. The suggestion of G. Meyer, to investigate and to exploit a nonlinear equivalent of the notion of steady-state response in order to obtain stable tracking on a nonminimum-phase system, has proved of fundamental importance and is gratefully acknowledged.

II. Notations and Problem Statement

Throughout this paper we consider a multivariable nonlinear system modeled by differential equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)w \tag{2.1a}$$

$$\dot{w} = s(w) \tag{2.1b}$$

$$e = h(x) + q(w). \tag{2.2}$$

The first equation of this system describes a *plant*, with state x, defined on a neighborhood X of the origin of $\boldsymbol{R}^n$, and input $u \in \boldsymbol{R}^m$, subject to the effect of a *disturbance* represented by the vector field $p(x)w$. The third equation defines the *error* $e \in \boldsymbol{R}^p$ between the actual plant output $h(x)$ and a *reference* signal $q(w)$ which the plant output is required to track. The second equation describes an autonomous system, the so-called *exosystem*, defined in a neighborhood W of the origin of $\boldsymbol{R}^s$, which models the class of disturbance and reference signals taken into consideration.

The vector $f(x)$, the m columns of the matrix $g(x)$, and the s columns of the matrix $p(x)$, are smooth (i.e., C^∞) vector fields on X; $s(x)$ is a smooth vector field on W; $h(x)$ and $q(w)$ are

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smooth mappings defined on X and, respectively, on W, with values in $\boldsymbol{R}^p$. We assume also that $f(0)=0$, $s(0)=0$, $h(0)=0$, $q(0)=0$. Thus, for $u=0$, the composite system (2.1) has an equilibrium state $(x, w)=(0, 0)$ with zero error (2.2).

The control action to (2.1) can be provided either by state feedback or by error feedback. A *state feedback controller* has the form

$$u=\alpha(x, w) \tag{2.3}$$

where $\alpha(x, w)$ is a C^k (for some integer $k \geq 2$) mapping defined on $X \times W$. Composing (2.1) with (2.3) yields a closed-loop system

$$\dot{x}=f(x)+g(x)\alpha(x, w)+p(x)w \tag{2.4a}$$

$$\dot{w}=s(w). \tag{2.4b}$$

For convenience, we assume $\alpha(0, 0)=0$, so that the closed loop (2.4) has an equilibrium at $(x, w)=(0, 0)$. An *error feedback controller* has the form

$$u=\theta(z) \tag{2.5a}$$

$$\dot{z}=\eta(z, e), \tag{2.5b}$$

i.e., is a dynamical system with state z, defined on a neighborhood Z of the origin of $\boldsymbol{R}^\nu$. For each $e \in \boldsymbol{R}^p$, $\eta(z, e)$ is a C^k vector field on Z, and $\theta(z)$ is a C^k mapping defined on Z (again, for some integer $k \geq 2$). The composition of (2.1) with (2.5) yields a closed-loop system

$$\dot{x}=f(x)+g(x)\theta(z)+p(x)w \tag{2.6a}$$

$$\dot{z}=\eta(z, h(x)+q(w)) \tag{2.6b}$$

$$\dot{w}=s(w). \tag{2.6c}$$

Again, we assume $\eta(0, 0)=0$ and $\theta(0)=0$, so that the triplet $(x, z, w)=(0, 0, 0)$ is an equilibrium of the closed loop (2.6).

The purpose of control action is to achieve local asymptotic stability and output regulation. *Local asymptotic stability* means that, *when the exosystem is disconnected* (i.e., when w is set equal to 0), the closed-loop system (2.4) [respectively, (2.6)] has an asymptotically stable equilibrium at $x=0$ [respectively, $(x, z)=(0, 0)$]. More precisely, it is required that the equilibrium in question be locally *exponentially* stable. *Output regulation* means that in the closed-loop system (2.4) [respectively, (2.6)], for all initial states $(x(0), w(0))$ [respectively, $(x(0), z(0), w(0))$] sufficiently close to the origin, $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

Formally, all of this can be summarized in the following two synthesis problems.

Problem 1 (State Feedback Regulator Problem): Find, if possible, $\alpha(x, w)$ such that:

1a) the equilibrium $x=0$ of

$$\dot{x}=f(x)+g(x)\alpha(x, 0)$$

is exponentially stable;

1b) there exists a neighborhood $U \subset X \times W$ of $(0,0)$ such that, for each initial condition $(x(0), w(0)) \in U$, the solution of (2.4) satisfies

$$\lim_{t \rightarrow \infty}(h(x(t))+q(w(t)))=0.$$

Problem 2 (Error Feedback Regulator Problem): Find, if possible, $\theta(z)$ and $\eta(z, e)$, such that:

2a) the equilibrium $(x, z)=(0, 0)$ of

$$\dot{x}=f(x)+g(x)\theta(z)$$

$$\dot{z}=\eta(z, h(x))$$

is exponentially stable;

2b) there exists a neighborhood $U \subset X \times Z \times W$ of $(0,0,0)$ such that, for each initial condition $(x(0), z(0), w(0)) \in U$, the solution of (2.6) satisfies

$$\lim_{t \rightarrow \infty}(h(x(t))+q(w(t)))=0.$$

III. Review of Linear Regulator Theory

For the sake of completeness, we briefly recall how Problems 1 and 2 have been solved for a linear multivariable system. In order to facilitate a comparison to the theory that will be developed in the next sections, it is convenient to regard the linear system in question as an approximation of the nonlinear system (2.1), (2.2). At the equilibrium $(x, w)=(0, 0)$, this approximation has the form (recall that $f(0)=0$, $s(0)=0$, $h(0)=0$, $q(0)=0$)

$$\dot{x}=Ax+Bu+Pw \tag{3.1a}$$

$$\dot{w}=Sw \tag{3.1b}$$

$$e=Cx+Qw \tag{3.2}$$

where A, B, P, S, C, Q are defined by

$$A=\left[\frac{\partial f}{\partial x}\right]_{x=0} \qquad B=g(0) \qquad P=p(0)$$

$$S=\left[\frac{\partial s}{\partial w}\right]_{w=0} \qquad C=\left[\frac{\partial h}{\partial x}\right]_{x=0} \qquad Q=\left[\frac{\partial q}{\partial w}\right]_{w=0}.$$

Accordingly, state feedback and error feedback controllers can be approximated as (recall that, by assumption, $\alpha(0, 0)=0$, $\theta(0)=0$, $\eta(0, 0)=0$)

$$u=Kx+Lw \tag{3.2}$$

and, respectively,

$$u=Hz \tag{3.3a}$$

$$\dot{z}=Fz+Ge \tag{3.3b}$$

with K, L, H, F, G defined by

$$K=\left[\frac{\partial \alpha}{\partial x}\right]_{x=0, w=0} \qquad L=\left[\frac{\partial \alpha}{\partial w}\right]_{x=0, w=0}$$

$$H=\left[\frac{\partial \theta}{\partial z}\right]_{z=0} \qquad F=\left[\frac{\partial \eta}{\partial z}\right]_{z=0, e=0} \qquad G=\left[\frac{\partial \eta}{\partial e}\right]_{z=0, e=0}.$$

The linear regulator theory (see [5]) is based the following three assumptions:

A1: $\sigma(S) \subset \operatorname{clos}(\boldsymbol{C}^+)=\{\lambda \in \boldsymbol{C}\operatorname{Re}[\lambda] \geq 0\}$;

A2: the pair (A, B) is stabilizable;

A3: the pair

$$[C \quad Q], \quad \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}$$

is detectable.

The first one of these does not involve a loss of generality because asymptotically stable modes in the exosystem do not affect the regulation of the output. The second one is indeed necessary for asymptotic stabilization of the closed loop via either state or error feedback. The third one is stronger than the assumption of detectability of the pair (C, A), that would be necessary for asymptotic stabilization of the closed loop via error feedback, but again does not involve loss of generality, as discussed in detail by Francis in [5]. In fact, if the pair (C, A) is detectable and A3 does not hold, it is always possible to reduce the dimension of the exosystem which actually affects the error, and have—on the reduced system thus obtained—condition A.3 satisfied.

The following results, due to Francis, describe necessary and sufficient conditions for the existence of solutions to Problems 1 and 2.

Proposition 1: Suppose A1 and A2 hold. Then, the linear state feedback regulator problem is solvable if and only if there exist matrices Π and Γ which solve the linear matrix equations

$$\Pi S = A\Pi + B\Gamma + P \tag{3.4a}$$

$$C\Pi + Q = 0. \tag{3.4b}$$

Proposition 2: Suppose A1, A2, and A3 hold. Then, the linear error feedback regulator problem is solvable if and only if there exist matrices Π and Γ which solve the linear matrix equations (3.4).

In summary, if assumptions A1, A2, and A3 hold, then the error feedback regulator problem is solvable if and only if the state feedback regulator problem is solvable, and the conditions for the existence of solutions can be expressed in terms of the solvability of certain linear matrix equations.

In [7], Hautus has proved that the possibility of solving these matrix equations can be characterized in terms of a comparison between the *transmission polynomials* of the system (3.1), (3.2) (in which u is considered as the input and e as the output) and those of the system

$$\dot{x} = Ax + Bu \tag{3.6a}$$

$$\dot{w} = Sw \tag{3.6b}$$

$$e = Cx. \tag{3.7}$$

The latter can be interpreted as the system obtained from (3.1), (3.2) by cutting the connections between the exosystem and the plant. In fact, Hautus proved the following result.

Proposition 3: The linear matrix equations (3.4) are solvable if and only if the system (3.1), (3.2) and the system (3.6), (3.7) have the same transmission polynomials.

IV. Standing Assumptions

We list hereafter three basic assumptions on which our approach to nonlinear regulator theory is based.

H1: $w = 0$ is a stable equilibrium of the exosystem, and there exists a neighborhood $\hat{W} \subset W$ of the origin with the property that each initial condition $w(0) \in \hat{W}$ is Poisson stable. Recall that in a nonlinear system of the form (2.1b) an initial condition w° is said to be Poisson stable if the flow $\Phi_t^s(w^\circ)$ of the vector field $s(w)$ is defined for all $t \in \boldsymbol{R}$, and for each neighborhood U of w° and for each real number $T > 0$, there exists a time $t_1 > T$ such that $\Phi_{t_1}^s(w^\circ) \in U$, and a time $t_2 < -T$ such that $\Phi_{t_2}^s(w^\circ) \in U$.

H2: The pair $f(x)$, $g(x)$ has a stabilizable linear approximation at $x = 0$.

H3: The pair

$$\begin{bmatrix} f(x) + p(x)w \\ s(w) \end{bmatrix}, \qquad h(x) + q(w)$$

has a detectable linear approximation at $(x, w) = (0, 0)$.

Remark 1: Hypothesis H1 implies that the matrix S, which characterizes the linear approximation of the exosystem at the equilibrium $w = 0$, has all its eigenvalues on the imaginary axis. In fact, no eigenvalue of S can have positive real part because otherwise the equilibrium would be unstable. Moreover, no eigenvalue of S can have negative real part either because otherwise the exosystem would admit a stable invariant manifold near the equilibrium, and trajectories originating on this manifold would converge to the origin as time tends to infinity. To some extent, hypothesis H1 has a similar, although stronger (because eigenvalues with positive real part are not allowed), role to that of the corresponding assumption A1 of linear regulator theory. As in the linear regulator theory, this assumption is instrumental in providing that certain conditions are necessary for the solvability of (either) regulator problems, and can be dispensed of if one is interested only in sufficient conditions. Note that the class of exosystems which satisfy assumption H1 includes the (very important in practice) class of systems in which every solution is a periodic solution. The two other hypotheses, H2 and H3, which in the case of linear systems reduce exactly to the corresponding assumptions A2 and A3, are obviously dictated by the requirement of achieving exponential stability [requirements 1a) and, respectively, 2a)]. ■

Remark 2: The closed-loop system (2.4) has a triangular structure

$$\dot{x} = \phi(x, w)$$

$$\dot{w} = s(w).$$

If the requirement 1a) is fulfilled, $x = 0$ is an asymptotically stable equilibrium of $\dot{x} = \phi(x, 0)$. Moreover, by hypothesis H1, $w = 0$ is a stable equilibrium of $\dot{w} = s(w)$. Thus, from well-known stability properties of triangular systems (see, e.g., Vidyasagar [16]), we can conclude that $(x, w) = (0, 0)$ is a *stable* equilibrium of (2.4). Similar considerations apply to the closed-loop system (2.6). In other words, requirement 1a) [respectively, requirement 2a)]—i.e., the *asymptotic stability* of the closed loop with no input coming from the exosystem—implies *stability* when the closed loop is driven by the exosystem. ■

V. Solution of Regulator Problems

We begin the discussion of the conditions for the solvability of Problem 1 by proving a useful lemma, which is a nonlinear extension of a result given in [6].

Lemma 1: Suppose H1 holds and assume that, for some $\alpha(x, w)$, condition 1a) is fulfilled. Then, condition 1b) is also fulfilled if and only if there exists a C^k $(k \geq 2)$ mapping $x = \pi(w)$, with $\pi(0) = 0$, defined in a neighborhood $W^\circ \subset W$ of 0, satisfying the conditions

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\alpha(\pi(w), w) + p(\pi(w))w \tag{5.1a}$$

$$h(\pi(w)) + q(w) = 0. \tag{5.1b}$$

Proof: The closed-loop system (2.4) has the form

$$\dot{x} = (A + BK)x + (P + BL)w + \phi(x, w)$$

$$\dot{w} = Sw + \psi(w)$$

where $\phi(x, w)$ and $\psi(w)$ vanish at the origin with their first-order derivatives. By assumption, the eigenvalues of the matrix $(A + BK)$ are in $\boldsymbol{C}^-$, and those of the matrix S on the imaginary axis. Thus, the system in question has a center manifold at $(0, 0)$, the graph of a C^k mapping

$$x = \pi(w)$$

with $\pi(w)$ satisfying (5.1a). This manifold is locally invariant under the flow of (2.4) and the restriction of the flow of (2.4) to this manifold is a diffeomorphic copy of the flow of the exosystem. Thus, by hypothesis H1, for every sufficiently small w°, the trajectory $(x(t), w(t))$ of (2.4) which satisfies $(x(0), w(0)) = (\pi(w^\circ), w^\circ)$, for every $\epsilon > 0$ and every $T > 0$

is such that

$$\|x(t) - \pi(w^\circ)\| < \epsilon, \qquad \|w(t) - w^\circ\| < \epsilon$$

at some $t > T$. Then 1a) can hold only if this center manifold is annihilated by the error map $e = h(x) + q(w)$, i.e., (5.1b) holds.

In order to prove the sufficiency, observe that, if (5.1a) is satisfied, the graph of the mapping $x = \pi(w)$ is a center manifold for (2.4). Moreover, by (5.1b),

$$e(t) = h(x(t)) - h(\pi(w(t))).$$

The equilibrium $(x, w) = (0, 0)$ of (2.4) is stable and the center manifold is locally attractive, i.e., satisfies (see the Appendix)

$$\|x(t) - \pi(w(t))\| \leq Me^{-at} \|x(0) - \pi(w(0))\| \quad (M > 0, a > 0)$$

for all $x(0)$, $w(0)$ sufficiently close to zero and all $t \geq 0$. Thus, by continuity of $h(x)$, $e(t) \to 0$ as $t \to \infty$, i.e., that condition 1b) is satisfied. ■

From this result it is easy to deduce a necessary and sufficient condition for the solution of the state feedback regulator problem.

Theorem 1: Under hypotheses H1 and H2, the state feedback regulator problem is solvable if and only if there exist C^k $(k \geq 2)$ mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying the conditions

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \quad (5.2a)$$

$$h(\pi(w)) + q(w) = 0. \quad (5.2b)$$

Proof: The necessity follows immediately from Lemma 1. For the sufficiency, observe that, by hypothesis H2, there exists a matrix K such that $(A + BK)$ has eigenvalues in $\mathbb{C}^-$. Suppose conditions (5.2) are satisfied for some $\pi(w)$ and $c(w)$, and set

$$\alpha(x, w) = c(w) + K(x - \pi(w)).$$

This choice clearly satisfies 1a). In fact, the Jacobian matrix of $f(x) + g(x)\alpha(x, 0)$ is exactly equal to $(A + BK)$. Moreover, by construction

$$\alpha(\pi(w), w) = c(w)$$

and therefore, (5.2a) reduces to (5.1a). On the other hand, (5.2b) is identical to (5.1b). Thus, by Lemma 1, requirement 1b) is also fulfilled. ■

Remark 3: Note, in the case of a linear system, the conditions thus found reduce *exactly* to the conditions described in Proposition 1. As a matter of fact, conditions (3.4) are exactly the conditions which the matrices Π and Γ should fulfill so that the linear mappings

$$x = \Pi w$$

$$u = \Gamma w$$

solve the corresponding linear version of (5.2). ■

Remark 4: The first one of the two conditions (5.2) expresses the fact that the graph of the mapping $x = \pi(w)$, which is a smooth manifold, is rendered locally *invariant* under the effect of a suitable feedback law, namely $u = c(w)$. The second condition expresses the fact that this manifold is *annihilated* by the error map $e = h(x) + q(w)$.

Remark 5: From the proof of the sufficiency in Theorem 1, we deduce that, if the mappings $\pi(w)$ and $c(w)$ which satisfy (5.2) are available, a controller solving the regulator problem can be obtained by choosing

$$\alpha(x, w) = c(w) + K(x - \pi(w))$$

where K is any matrix which places the eigenvalues of $(A + BK)$ in $\mathbb{C}^-$. Note that the controller thus defined has the form $\alpha(x, w) = Kx + \beta(w)$. One may interpret the result thus established in the following way. The state feedback $u = Kx + v$ renders the system

$$\dot{x} = f(x) + g(x)Kx + p(x)w + g(x)v$$

$$y = h(x)$$

exponentially stable. On this system, the input $v(t) = \beta(w(t))$, with $\beta(w) = c(w) - K\pi(w)$, imposes a response which (exponentially) converges to a steady-state response exactly equal to $y_{ss}(t) = -q(w(t))$. ■

We proceed now with the solution of the regulator problem via error feedback.

Lemma 2: Suppose H1 holds and assume that, for some $\theta(z)$ and $\eta(z, e)$, condition 2a) is fulfilled. Then, condition 2b) is also fulfilled if and only if there exist C^k $(k \geq 2)$ mappings $x = \pi(w)$, with $\pi(0) = 0$, and $z = \sigma(w)$, with $\sigma(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying the conditions

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\theta(\sigma(w)) + p(\pi(w))w \quad (5.3a)$$

$$\frac{\partial \sigma}{\partial w} s(w) = \eta(\sigma(w), 0) \quad (5.3b)$$

$$h(\pi(w)) + q(w) = 0. \quad (5.3c)$$

Proof: The closed-loop system (2.6) has the form

$$\dot{x} = Ax + BHz + Pw + \phi(x, z, w)$$

$$\dot{z} = GCx + Fz + GQw + \chi(x, z, w)$$

$$\dot{w} = Sw + \psi(w)$$

where $\phi(x, z, w)$, $\chi(x, z, w)$, and $\psi(w)$ vanish at the origin with their first-order derivatives. By assumption, the eigenvalues of the matrix

$$\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$$

are in $\mathbb{C}^-$, and those of the matrix S on the imaginary axis. Thus, the system has a center manifold at $(0, 0, 0)$, the graph of a mapping

$$x = \pi(w)$$

$$z = \sigma(w)$$

with $\pi(w)$ and $\sigma(w)$ satisfying (5.3a). By hypothesis H1, no trajectory on this manifold converges to 0. Then 2b) holds only if this manifold is annihilated by the error map $e = h(x) + q(w)$, i.e., (5.3c) holds. On the other hand, since the center manifold is locally attractive, the fulfillment of (5.3c) guarantees that $e(t) \to 0$ as $t \to \infty$, i.e., that condition 2b) is satisfied. ■

Remark 6: Note that condition (5.3b) expresses the fact that the feedback which solves the regulator problem incorporates an *internal model* of the exosystem (the fact that any error feedback solving the regulator problem necessarily incorporates an internal model of the exosystem was shown, in more general terms, by Hepburn and Wonham in [8]). ■

From this result it is easy to deduce a necessary and sufficient condition for the solution of the error feedback regulator problem.

Theorem 2: Under hypotheses H1, H2, and H3, the error feedback regulator problem is solvable if and only if there exists C^k ($k \geq 2$) mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying the conditions

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \quad (5.2a)$$

$$h(\pi(w)) + q(w) = 0. \quad (5.2b)$$

Proof: The necessity follows trivially from the previous lemma. The proof of sufficiency is constructive, as in Theorem 1. By assumption, there exist matrices H, G_1, G_2 such that

$$A + BH \quad \text{and} \quad \begin{bmatrix} A - G_1C & P - G_1Q \\ -G_2C & S - G_2Q \end{bmatrix}$$

have eigenvalues in $\boldsymbol{C}^-$. A standard calculation also shows that

$$\begin{bmatrix} A & BH & BK \\ G_1C & A + BH - G_1C & P + BK - G_1Q \\ G_2C & -G_2C & S - G_2Q \end{bmatrix} \quad (5.4)$$

has eigenvalues in $\boldsymbol{C}^-$ for any K.

Suppose conditions (5.2) are satisfied for some $\pi(w)$ and $c(w)$, and consider a feedback

$$u = \theta(z)$$

$$\dot{z} = \eta(z, e)$$

with z, $\theta(z)$, and $\eta(z)$, defined in the following way:

$$z = \mathrm{col}(z_1, z_2)$$

$$\theta(z) = c(z_2) + H(z_1 - \pi(z_2))$$

$$\eta(z_1, z_2, e) = \mathrm{col}(\eta_1(z_1, z_2, e), \eta_2(z_1, z_2, e))$$

$$\eta_1(z_1, z_2, e) = f(z_1) + p(z_1)z_2 + g(z_1)(c(z_2) + Hz_1 - H\pi(z_2)) - G_1(h(z_1) + q(z_2) - e)$$

$$\eta_2(z_1, z_2, e) = s(z_2) - G_2(h(z_1) + q(z_2) - e).$$

The Jacobian matrix of

$$\dot{x} = f(x) + g(x)\theta(z)$$

$$\dot{z} = \eta(z, h(x))$$

has exactly the form (5.4), where

$$K = \left[\frac{\partial c}{\partial w}\right]_{w=0} - H\left[\frac{\partial \pi}{\partial w}\right]_{w=0}.$$

Thus, requirement 2b) is satisfied. A straightforward calculation shows that, by construction, conditions (5.2) imply the fulfillment of (5.3), with

$$\sigma(w) = \mathrm{col}(\pi(w), w).$$

Thus, requirement 2b) follows from Lemma 2. ■

Note that, as in the linear regulator theory, the conditions for the solution of the state feedback regulator problem and those for the solution of the error feedback regulator problem are identical. The proof of Theorem 2 also shows how, if the mappings $\pi(w)$ and $c(w)$ which satisfy (5.2) are available, a controller solving the error feedback regulator problem can be easily constructed.

VI. On the Nonlinear Analog of the Notion of Transmission Zeros

The purpose of the next three sections is to show that, in the equations (5.2), the only true unknown is the mapping $\pi(w)$ whose graph defines a controlled invariant manifold, in the sense that the determination of the auxiliary mapping $c(w)$, which actually renders locally invariant the manifold in question, does not pose any additional problem. This will be seen to follow from a nonlinear counterpart of the result, proved by Hautus and discussed at the end of Section III, which relates the solvability of regulator problems to system zeros. To this end, we need to summarize briefly some results about the nonlinear analog of the notion of transmission zeros; for more detailed discussions and additional results, the reader is referred, for instance, to [10, pp. 289–310].

Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u \quad (6.1a)$$

$$y = h(x) \quad (6.1b)$$

defined on a neighborhood X of the origin of $\boldsymbol{R}^n$, in which, as before, we suppose $f(0) = 0$ and $h(0) = 0$. Let M be a smooth connected submanifold of x which contains the point $x = 0$. The manifold M is said to be *locally controlled invariant* if there exist a smooth mapping u: $M \to \boldsymbol{R}^m$, and a neighborhood U of the origin in $\boldsymbol{R}^n$, such that the vector field $\tilde{f}(x) = f(x) + g(x)u(x)$ is *tangent* to M for all $x \in M \cap U$.

An *output zeroing submanifold* is a connected submanifold M of X which contains the origin and satisfies the following:

i) M is locally controlled invariant;

ii) $h(x) = 0$ for all $x \in M$.

In other words, an output zeroing submanifold is a submanifold M of the state space with the property that—for some choice of feedback control $u(x)$—the trajectories of the closed-loop system

$$\dot{x} = f(x) + g(x)u(x)$$

$$y = h(x)$$

which start in M stay in M for some interval of time, and the corresponding output is identically zero in the meanwhile.

If M and M' are connected smooth submanifolds of X which both contain the point $x = 0$, we say that M *locally contains* M' (or, *coincides with* M') if for some neighborhood U of the origin, $M \cap U \supset M' \cap U$ (or $M \cap U = M' \cap U$). An output zeroing submanifold M is *locally maximal* if, for some neighborhood U of the origin, any other output zeroing submanifold M' satisfies $M \cap U \supset M' \cap U$. The construction of a locally maximal output zeroing submanifold is illustrated in the following statement.

Proposition 4:

Part 1: Define a nested sequence of subsets $M_0 \supset M_1 \supset \cdots$ of X in the following way. Set $M_0 = \{x \in X : h(x) = 0\}$. At each $k > 0$, suppose that, for some neighborhood U_{k-1} of 0, $M_{k-1} \cap U_{k-1}$ is a smooth manifold, let $\tilde{M}_{k-1}$ denote the connected component of $M_{k-1} \cap U_{k-1}$ which contains the origin ($\tilde{M}_{k-1}$ is nonempty because $f(0) = 0$) and define M_k as

$$M_k = \{x \in \tilde{M}_{k-1}:\ f(x) \in \mathrm{span}\,\{g_1(x), \cdots, g_m(x)\} + T_x\tilde{M}_{k-1})\}.$$

Then, for some $k^* \geq 0$ and some neighborhood U_{k^*} of 0, $M_{k^*+1} = \tilde{M}_{k^*}$. Suppose also that the subspaces span $\{g_1(x), \cdots, g_m(x)\}$ and span $\{g_1(x), \cdots, g_m(x)\} \cap T_x\tilde{M}_{k^*}$ have constant dimensions for all $x \in M_k$. Then, the manifold $Z^* = \tilde{M}_{k^*}$ is a locally maximal output zeroing submanifold.

Part 2: If, in addition,

$$\dim(\mathrm{span}\,\{g_1(x), \cdots, g_m(x)\}) = m, \quad (6.2a)$$

$$\mathrm{span}\,\{g_1(x), \cdots, g_m(x)\} \cap T_xZ^* = 0 \quad (6.2b)$$

at $x = 0$, then there exists a *unique* smooth mapping $u^*: Z^* \to \boldsymbol{R}^m$ such that the vector field

$$f^*(x) = f(x) + g(x)u^*(x)$$

is tangent to Z^*. ∎

Suppose the hypotheses listed in the previous Proposition are satisfied. Since the vector field $f^*(x)$ is tangent to Z^*, the *restriction* of $f^*(x)$ to Z^* is a well-defined vector field of Z^*. In what follows, unless otherwise specified, by $f^*(x)$ we will always indicate—with some abuse of notation—the restriction of $f^*(x)$ to Z^*. The submanifold Z^* is called the (local) *zero dynamics submanifold* and the vector field $f^*(x)$ is called the *zero dynamics vector field*. The pair (Z^*, f^*) is called the *zero dynamics* of the system (6.1).

By construction, the dynamical system

$$\dot{x} = f^*(x), \qquad x \in Z^* \tag{6.3}$$

identifies the internal dynamical behavior induced on the system when the output has been forced, by proper choice of initial state and input, to remain zero for some interval of time.

Remark 7: In the case of a linear system, all the hypotheses of Proposition 4, Part 1, which lead to the existence of the manifold Z^*, are always satisfied. The latter coincides with the largest *controlled invariant subspace*, noted V^*, of $\ker(C)$, namely the largest subspace of $\ker(C)$ satisfying

$$AV^* \subset V^* + \operatorname{Im}(B).$$

The hypotheses (6.2) of Proposition 4, Part 2, reduce to the following ones:

$$\dim(\operatorname{Im}(B)) = m, \qquad V^* \cap \operatorname{Im}(B) = 0 \tag{6.4}$$

which, as is well known, are exactly the conditions under which the transfer function matrix of the system is left invertible (in particular, the second one is equivalent to the condition that the *largest controllability subspace*, noted R^*, of $\ker(C)$ is zero [17]). The state feedback $u^*(x)$ which renders $f^*(x)$ tangent to V^* is a linear function of x, namely $u^*(x) = Fx$, and by construction the subspace V^* is *invariant* under the linear mapping $(A + BF)$. The restriction $(A + BF)|_{V^*}$ identifies a linear autonomous system, defined on V^*. In case assumptions (6.4) hold, the restriction $F|_{V^*}$ is unique and, if the triple (A, B, C) is minimal, the invariant factors of $(A + BF)|_{V^*}$ are the *transmission polynomials* and their roots are the *transmission zeros* of the system.

VII. Solvability of Regulator Problems is a Property of the Zero Dynamics

The relevance of the concepts summarized in the previous section can be understood in the following way. Consider again the system (2.1), (2.2), which, for convenience, will be denoted by Σ_e and rewritten in compact form as

$$\dot{x}_e = f_e(x_e) + g_e(x_e)u$$

$$e = h_e(x_e)$$

having set

$$x_e = \operatorname{col}(x, w),$$

$$f_e(x_e) = \operatorname{col}(f(x) + p(x)w, s(w))$$

$$g_e(x_e) = \operatorname{col}(g(x), 0)$$

$$h_e(x_e) = h(x) + q(w).$$

Suppose that conditions (5.2), whose fulfillment was proved to be necessary and sufficient for the existence of a solution to the (either state feedback or error feedback, as shown in Section V) regulator problem, are satisfied, and consider, in the state-space $X_e = X \times W$ of Σ_e, the submanifold

$$M_s = \{(x, w) \in X_e: \quad x = \pi(w)\}. \tag{7.1}$$

In view of the terminology introduced above, we may easily observe that the manifold thus defined is an *output zeroing submanifold* of the system Σ_e. In fact, (5.2a) exactly says that M_s is locally controlled invariant [invariance is achieved under the feedback law $u = c(w)$], and (5.2b) says that M_s is annihilated by the "output" map $e = h_e(x_e)$.

The system in question may have also another relevant output zeroing submanifold. Let Σ denote the system

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

(with $f(x)$, $g(x)$, and $h(x)$ the same as in (2.1), (2.2), that is in Σ_e) and suppose Σ satisfies the assumptions of Proposition 4, so that a zero dynamics (Z^*, f^*) can be defined in a neighborhood $U \subset X$ of 0. Let $u^*(x)$ denote the (unique) feedback law which renders $f^*(x) = f(x) + g(x)u^*(x)$ tangent to Z^*, and consider, in the state-space $X_e = X \times W$ of Σ_e, the submanifold

$$M_z = Z^* \times \{0\}. \tag{7.2}$$

This submanifold is indeed an output zeroing submanifold of Σ_e. For this manifold is locally controlled invariant [invariance is achieved under the feedback law $u = u^*(x)$ because $w = 0$ is an equilibrium of $\dot{w} = s(w)$] and is also annihilated by the output map (because $h(x) = 0$ for each $x \in Z^*$, and $q(0) = 0$).

In the next statement, we illustrate more precisely the relation between the zero dynamics of Σ_e and that of Σ. Of course, our analysis requires the existence of both the zero dynamics in question, so we assume

Z1: the system Σ satisfies the hypotheses of Proposition 4, so that a zero dynamics (Z^*, f^*) can be defined in a neighborhood $U \subset X$ of 0;

Z2: the system Σ_e satisfies the hypotheses of the (corresponding) Proposition 4, so that a zero dynamics (Z_e^*, f_e^*) can be defined in a neighborhood $U_e \subset X_e$ of 0.

Lemma 3: Suppose hypotheses Z1 and Z2 hold. Suppose that, in a neighborhood of $x_e = 0$, the set

$$M = Z_e^* \cap (X \times \{0\}) \tag{7.3}$$

is a smooth submanifold. Then M locally coincides with the submanifold M_z defined by (7.2). Moreover, M is locally invariant under f_e^*, and the restriction of f_e^* to M is locally diffeomorphic to the vector field $f^*(x)$ which characterizes the zero dynamics of Σ.

Proof: Since Z_e^* is a locally maximal output zeroing submanifold for Σ_e, Z_e^* locally contains the submanifold M_z, which is an output zeroing submanifold for Σ_e. Since by construction $M_z \subset (X \times \{0\})$, we deduce that M locally contains M_z. Observe now that, at each $(x, 0) \in M$, $h(x) = 0$. In fact, $h(x) + q(w) = 0$ at each $(x, w) \in Z_e^*$, and $w = 0$ implies $q(w) = 0$. Moreover, if we let $u_e^*(x, w)$ denote the (unique) input which constrains the flow of Σ_e to evolve on Z_e^*, we immediately see that the input $u = u_e^*(x, 0)$ constrains the flow of (6.1) to evolve on M [because $w = 0$ is an equilibrium of (2.1b)]. From this we deduce that M is an output zeroing submanifold for Σ. M must be locally contained in M_z, and therefore M locally coincides with M_z.

The second part of the statement is proved in this way. At each point $(x, 0)$ of M_z, the input $u^*(x)$ renders the vector $f(x) + g(x)u^*(x)$ tangent to M_z. Σ_e satisfies the assumptions of Proposition 4 [in particular (6.2)] and, at each $x_e \in Z_e^*$ there is a *unique* u_e^* such that $f_e^* = f_e + g_e u_e^*$ is tangent to Z_e^*. Since

$M_z \subset Z_e^*$, we deduce that $u^*(x) = u_e^*(x, 0)$ for each $x \in M_z$. The immersion

$$\sigma: \quad Z^* \to X_e$$

$$x \to (x, 0)$$

is diffeomorphism of Z^* onto M_z, and

$$\sigma_* f^*(x) = (f_e + g_e u_e^*) \circ \sigma(x) = (f_e^*) \circ \sigma(x).$$

Thus, the restriction of f_e^* to M_z is locally diffeomorphic to the vector field $f^*(x)$. ∎

The next lemma illustrates, in terms of properties of the zero dynamics of Σ_e, necessary conditions for the solvability of the regulator equations (5.2).

Lemma 4: Suppose hypotheses Z1 and Z2 hold. Suppose there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying conditions (5.2). Then:

i) In a neighborhood of $x_e = 0$, the set M defined by (7.3) is a smooth submanifold.

ii) Z_e^* locally contains the submanifold M_s, defined by (7.1), and

$$T_0 Z_e^* = T_0 M_s \oplus T_0 M.$$

iii) M_s is locally invariant under f_e^*, and the restriction of f_e^* to Z_s is locally diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

Proof: Since Z_e^* is a locally maximal output zeroing submanifold for Σ_e, Z_e^* locally contains M_z and M_s, which are output zeroing submanifolds for Σ_e. At $(x, w) = (0, 0)$, $T_0 M_s \oplus T_0(X \times \{0\}) = T_0 X_e$. Thus, since $T_0 M_s \subset T_0 Z_e^*$, we have $T_0 Z_e^* + T_0(X \times \{0\}) = T_0 X_e$. By continuity, $T_{(x,w)} Z_e^*$ and $T_{(x,w)}(X \times \{0\})$ span $T_{(x,w)} X_e$ for all $(x, w) \in Z_e^* \cap (X \times \{0\})$ in a neighborhood of (0, 0) and this implies i).

By definition of M,

$$\dim(Z_e^*) \leq s + \dim(M)$$

and therefore, since $s = \dim(M_s)$, we obtain

$$\dim(Z_e^*) \leq \dim(M_s) + \dim(M).$$

Also the reverse inequality holds because $T_0 M_s \cap T_0 M = \{0\}$, and this proves ii).

By (5.2a), at each point $(\pi(w), w)$ of M_s, the input $u = c(w)$ renders the vector

$$f_e(\pi(w), w) + g_e(\pi(w), w) c(w)$$

tangent to M_s. Thus, again appealing to the uniqueness of u_e^*, we deduce that $c(w)$ necessarily coincides with $u_e^*(\pi(w), w)$. The immersion

$$\mu: \quad W^\circ \to X_e$$

$$w \to (\pi(w), w)$$

induces a local diffeomorphism of a neighborhood of the origin in W° onto a neighborhood of the origin in M_s. Again by (5.2) [with $c(w)$ replaced by $u_e^*(\pi(w), w)$] we have

$$\mu_* s(w) = (f_e + g_e u_e^*) \circ \mu(w) = (f_e^*) \circ \mu(w).$$

Thus, the restriction of f_e^* to M_s is locally diffeomorphic to the vector field $s(w)$, i.e., iii) holds. ∎

We are now in a position to prove a nonlinear analog of Hautus' result.

Theorem 3: Suppose hypotheses Z1, Z2 hold. Let (Z_e^*, f_e^*) denote the zero dynamics of Σ_e. Then there exist smooth mappings $x = \pi(w)$, with $\pi(0) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying conditions (5.2), if and only if the zero dynamics of Σ_e have the following properties.

i) In a neighborhood of $x_e = 0$, the set M defined by (7.3) is a smooth submanifold.

ii) There exists a submanifold Z_s of Z_e^*, of dimension s, which contains the origin, such that

$$T_0 Z_e^* = T_0 Z_s \oplus T_0 M.$$

iii) Z_s is locally invariant under f_e^*, and the restriction of f_e^* to Z_s is locally diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

Proof: The necessity follows immediately from Lemma 4. In order to prove the sufficiency, we show first that Z_s can be locally expressed in the form of the graph of a smooth mapping $x = \gamma(w)$ defined in a neighborhood $W^\circ \subset W$ of 0. Note that, by definition, $T_0 Z_e^* \cap T_0(X \times \{0\}) = T_0 M$, and therefore ii) implies

$$T_0 Z_s \cap T_0(X \times \{0\}) = \{0\}. \tag{7.4}$$

By assumption, Z_s is an s-dimensional submanifold; thus, there is a smooth mapping

$$\phi: \tilde{X}_e \to \boldsymbol{R}^n$$

(where $\tilde{X}_e$ is a neighborhood of 0 in X_e), such that

$$Z_s = \{(x, w) \in \tilde{X}_e: \; \phi(x, w) = 0\}.$$

Condition (7.4) implies that the matrix $(\partial\phi/\partial x)$ is invertible at 0 and, by the implicit function theorem, there exists a smooth mapping $x = \gamma(w)$ whose graph coincides with Z_s in a neighborhood of 0.

We prove now that iii) implies the fulfillment of conditions (5.2). As in the proof of Lemma 4, consider the immersion

$$\mu: \quad W^\circ \to X_e$$

$$w \to (\gamma(w), w)$$

which induces a local diffeomorphism of a neighborhood of the origin in W° onto a neighborhood of the origin in Z_s. By iii), f_e^* is tangent to Z_s, and there is a diffeomorphism ψ: $W^\circ \to W^\circ$, such that

$$(\mu \circ \psi)_* s(w) = (f_e^*) \circ \mu \circ \psi(w) = (f_e + g_e u_e^*) \circ \mu \circ \psi(w).$$

This clearly shows that the mappings

$$\pi(w) = \gamma(w)$$

$$c(w) = u_e^*(\gamma(w), w)$$

satisfy (5.2a). The fulfillment of (5.2b) is a straightforward consequence of the fact that Z_s is a submanifold of Z_e^* and the latter is annihilated by $h_e(x_e)$. ∎

Remark 8: Note that condition i), by Lemma 3, implies that M is locally invariant under f_e^*, and the restriction of f_e^* to M is locally diffeomorphic to the vector field $f^*(x)$ which characterizes the zero dynamics of Σ. Thus, (5.2) are solvable if and only if the zero dynamics of Σ_e possesses two complementary invariant submanifolds, the flows of f_e^* on these two submanifolds being diffeomorphic to that of the exosystem and, respectively, to that of the zero dynamics of Σ. This is exactly what is asserted, in the case of a linear system, by Hautus' result summarized in Proposition 3. ∎

The previous theorem shows clearly that the solvability of the regulator problem is a property of the zero dynamics of Σ_e. In particular, we stress the fact that the proof of this theorem demonstrates the existence of a smooth mapping $x = \pi(w)$, which solves (5.2) together with the mapping

$$c(w) = u_e^*(\pi(w), w) \tag{7.5}$$

where u_e^* is the unique input which renders f_e^* tangent to Z_e^*. In other words, if the zero dynamics of Σ_e are known (and so is the input u_e^*), the only real problem to deal with—in order to be able to solve a regulator problem—is the one of finding a submanifold Z_s of Z_e^* with the following properties:

- Z_s has dimension s;
- Z_s is transverse to $X \times \{0\}$;
- Z_s is locally invariant under the zero dynamics vector field f_e^*;
- the restriction of f_e^* to Z_s is diffeomorphic to the vector field $s(w)$ which characterizes the exosystem (2.1b).

We present hereafter a very expressive sufficient condition for the existence of such a submanifold.

Corollary 1: Suppose hypotheses H1, H2, H3 hold. Suppose also hypotheses Z1, Z2 hold. Let (Z^*, f^*) denote the zero dynamics of Σ, and (Z_e^*, f_e^*) the zero dynamics of Σ_e. The (either state or error) feedback regulator problem is solvable if:

i) $T_0Z_e^* + T_0(X \times \{0\}) = T_0X_e$;

ii) the zero dynamics of Σ have a hyperbolic equilibrium at $x = 0$.

Proof: As already observed in the proof of Lemma 4, i) implies that in a neighborhood of the origin, M is a smooth submanifold. Thus, by Lemma 3, M locally coincides with $M_z = Z^* \times \{0\}$. As a consequence, the vector field

$$f_e^*(x, w) = \mathrm{col}\,(f(x) + p(x)w + g(x)u_e^*(x, w), s(w))$$

at each point $(x, 0)$ of M coincides with the vector field $\mathrm{col}\,(f^*(x), 0)$, where $f^*(x)$ is the zero dynamics vector field of Σ. Set

$$A = \left[\frac{\partial f^*(x)}{\partial x}\right]_{x=0} \qquad A_e = \left[\frac{\partial f_e^*(x, w)}{\partial x} \frac{\partial f_e^*(x, w)}{\partial w}\right]_{x=0,\, w=0}$$

and note that

$$A_e = \begin{bmatrix} A & * \\ 0 & S \end{bmatrix}.$$

The restriction A^* of A to its invariant subspace T_0Z^* characterizes the linear approximation of the zero dynamics of Σ, and the restriction A_e^* of A_e to its invariant subspace $T_0Z_e^*$ characterizes the linear approximation of the zero dynamics of Σ_e. Note that $T_0(Z^* \times \{0\})$ is a subspace of $T_0Z_e^*$, with codimension s. Thus, the spectrum of A_e^* is the disjoint union of the spectra of A^* and of S. By assumption ii), A^* has no eigenvalue on the imaginary axis, while all eigenvalues of S are on the imaginary axis. From the center manifold theorem, we deduce that there is a C^k $(k \geq 2)$ center manifold Z_s for Z_e^*, which indeed satisfies conditions ii) and iii) of Theorem 3. This manifold can be expressed as a graph of a mapping $x = \pi(w)$, which satisfies (5.2) for $c(w) = u_e^*(\pi(w), w)$. ■

VIII. Application: Output Tracking in a Single-Input Single-Output System

We discuss now, in a simple example, how the conditions developed in the previous sections can be used in the particular case in which the system (2.1), (2.2) is single-input single-output and the vector $p(x)$ is zero, i.e., it is only required to track a desired reference output $y_R(t) = -q(w(t))$.

Let the system Σ have relative degree r at $x = 0$ (i.e., suppose $L_gL_f^kh(x) = 0$ for all $k \leq r-2$ and all x near 0, and $L_gL_f^{r-1}(h(0) \neq 0)$). An immediate calculation shows that also the system Σ_e has relative degree r. Consider now the (standard) change of coordinates $(z, \xi) = \varphi(x)$, with

$$\xi = \mathrm{col}\,(h(x), L_fh(x), \cdots, L_f^{r-1}h(x))$$

and z such that $L_gz = 0$. Under this change of coordinates the system (2.1), (2.2) becomes

$$\dot{z} = f_0(z, \xi)$$
$$\dot{w} = s(w)$$
$$\dot{\xi}_1 = \xi_2$$
$$\cdots$$
$$\dot{\xi}_{r-1} = \xi_r$$
$$\dot{\xi}_r = b(z, \xi) + a(z, \xi)u$$
$$e = \xi_1 + q(w)$$

where

$$b(z, \xi) = L_f^rh\circ\varphi^{-1}(z, \xi), \qquad a(z, \xi) = L_gL_f^{r-1}h\circ\varphi^{-1}(z, \xi).$$

Moreover, the equation

$$\dot{z} = f_0(z, 0)$$

characterizes the zero dynamics of Σ.

The identification of the zero dynamics of Σ_e is also very easy. In fact, imposing $e = 0$ yields

$$\xi_k = -L_s^{k-1}q(w)$$

for all $1 \leq k \leq r$. Therefore, the zero dynamics of Σ_e are characterized by the equations

$$\dot{z} = f_0(z, k(w))$$
$$\dot{w} = s(w)$$

in which

$$k(w) = -\mathrm{col}\,(q(w), L_sq(w), \cdots, L_s^{r-1}q(w)) \tag{8.1}$$

and the input $u_e^*(x, w)$ is given by

$$u_e^*(x, w) = -\frac{L_f^rh(x) + L_s^rq(w)}{L_gL_f^{r-1}h(x)}.$$

Clearly, both Σ and Σ_e satisfy the assumptions Z1 and Z2, so that the result of Theorem 3 can be used. Conditions ii) and iii) of Theorem 3 are satisfied if and only if there exists a mapping $z = \lambda(w)$, such that

$$\frac{\partial\lambda}{\partial w}s(w) = f_0(\lambda(w), k(w)) \tag{8.2}$$

and such a mapping exists, as shown by the Corollary to Theorem 3, if $\dot{z} = f_0(z, 0)$ has a hyperbolic equilibrium at $z = 0$.

Note that the mapping

$$\pi(w) = \mathrm{col}\,(\lambda(w), k(w)) \tag{8.3}$$

satisfies conditions (5.2), for

$$c(w) = u_e^*(\pi(w), w). \tag{8.4}$$

In summary (see Remark 5) we conclude that, if the zero dynamics of the plant has a hyperbolic equilibrium at $z = 0$, the state feedback regulator problem is solvable, by means of a feedback that, in the (z, ξ) coordinates, assumes the form

$$\alpha(z, \xi, w) = -\frac{b(\lambda(w), k(w)) + L_s^rq(w)}{a(\lambda(w), k(w))} + K_1(z - \lambda(w)) + K_2(\xi - k(w))$$

with $\lambda(w)$ a solution of the equation (8.2), $k(w)$ given by (8.1), and $K = (K_1 \;\; K_2)$ a $1 \times n$ row vector of real numbers such that the state feedback $u = K_1 z + K_2 \xi$ exponentially stabilizes the plant. In the same way, the reader will have no difficulty in deriving the description of an error feedback regulator, using the expression (8.3) and (8.4) found for $\pi(w)$ and $c(w)$ in the expressions of $\theta(z)$ and $\eta(z, e)$ illustrated in the proof of Theorem 2.

Remark 9: Note that, in the case of a linear system, equation (8.2) reduces to an equation of the form

$$\Lambda S = Q\Lambda + PK$$

in the unknown Λ, where Q is a matrix whose eigenvalues coincide with the zeros of the transfer function of the system. It is well known that this equation is universally solvable (i.e., solvable for all PK) if and only if the spectra $\sigma(S)$ and $\sigma(Q)$ are disjoint. Thus, one may interpret condition (8.2) as the nonlinear analog of the well-known condition according to which a linear regulator problem is solvable if no zero of the transfer function of the plant coincides with an eigenvalue of the exosystem. ■

IX. Concluding Remarks

It has been shown that the problem of achieving output regulation in a locally exponentially stable nonlinear closed-loop system is equivalent, under the reasonable assumption that the signals generated by the so-called exosystem are bounded but not trivially decaying to zero, to the problem of finding a locally controlled invariant manifold, for the dynamics of a composite system which incorporates the plant and the exosystem, and whose output is equal to the tracking error. This controlled invariant manifold is in fact a submanifold of the zero dynamics manifold of the composite system, and the flow of the zero dynamics on this invariant manifold is a diffeomorphic copy of the flow of the exosystem. In particular, it has been shown that a solution of the regulator problem exists always under the mild assumption that the zero dynamics of the plant have a hyperbolic equilibrium. The results constitute a very natural extension to the nonlinear setting of the results on multivariable linear regulators obtained by Francis [5], and of their interpretation in terms of system zeros discussed by Hautus [7].

Appendix

We summarize in this Appendix some relevant facts about center manifold theory; for a more detailed analysis the reader is referred, e.g., to Carr [3]. Consider a nonlinear system

$$\dot{x} = f(x) \tag{A.1}$$

where f is a C^r vector field ($r \geq 2$) defined on an open subset U of $\boldsymbol{R}^n$, and let $x^\circ \in U$ be a point of *equilibrium* for f, i.e., a point such that $f(x^\circ) = 0$. Without loss of generality, suppose $x^\circ = 0$. Let

$$F = \left[\frac{\partial f}{\partial x}\right]_{x=0}$$

denote the Jacobian matrix of f at $x = 0$.

Suppose the matrix F has n° eigenvalues with *zero* real part, n^- eigenvalues with *negative* real part, and n^+ eigenvalues with *positive* real part. Then, it is well known from linear algebra that the domain of the linear mapping F can be decomposed into the direct sum of three invariant subspaces, noted E°, E^-, E^+, of dimension n°, n^-, n^+, respectively. If the linear mapping F is viewed as a representation of the differential (at $x = 0$) of the nonlinear mapping f: $x \in U \to f(x) \in \boldsymbol{R}^n$, its domain is the tangent space T_0U to U at $x = 0$, and the three subspaces in question can be viewed as subspaces of T_0U satisfying

$$T_0U = E^\circ \dotplus E^- \dotplus E^+.$$

Definition: A C^r submanifold S of U is said to be *locally invariant* for (A.1), if for each $x^\circ \in S$, there exists $t_1 < 0 < t_2$ such that the integral curve $x(t)$ of (A.1) satisfying $x(0) = x^\circ$ is such that $x(t) \in S$ for all $t \in (t_1, t_2)$. ■

Definition: Let $x = 0$ be an equilibrium of (A.1). A manifold S, passing through $x = 0$, is said to be a *center manifold* for (A.1) at $x = 0$, if it is locally invariant and the tangent space to S at 0 is exactly E°. ■

Suppose all the eigenvalues of F have nonpositive real part, i.e., that $n^+ = 0$ [otherwise, the point $x = 0$ would be an unstable equilibrium of (A.1)]. One can always choose coordinates in U such that the system (A.1) is represented in the form

$$\dot{y} = Ay + g(y, z) \tag{A.2a}$$

$$\dot{z} = Bz + f(y, z) \tag{A.2b}$$

where A is an $(n^- \times n^-)$ matrix having all eigenvalues with negative real part, B is an $(n^\circ \times n^\circ)$ matrix having all eigenvalues with zero real part, and the functions g and f are C^r functions vanishing at $(y, z) = (0, 0)$ together with all their first-order derivatives. Thus, there is no loss of generality in assuming that the system (A.1) is given the form (A.2). Existence of center manifolds for a system of the form (A.2) is illustrated in the following statement.

Theorem: There exist a neighborhood $V \subset \boldsymbol{R}^{n^0}$ of $z = 0$ and a C^{r-1} mapping π: $V \to \boldsymbol{R}^{n^-}$ such that

$$S = \{(y, z) \in (\boldsymbol{R}^{n^-}) \times V: y = \pi(z)\}$$

is a center manifold for (A.2). ■

By definition, a center manifold for the system (A.2) passes through $(0, 0)$ and is tangent to the subset of points whose y coordinate is equal to 0. Thus, the mapping π satisfies

$$\pi(0) = 0, \qquad \frac{\partial \pi}{\partial z}(0) = 0. \tag{A.3a}$$

Moreover, this manifold is locally invariant for (A.2), and this imposes on the mapping π the constraint

$$\frac{\partial \pi}{\partial z}(Bz + f(\pi(z), z)) = A\pi(z) + g(\pi(z), z) \tag{A.3b}$$

as easily deduced by differentiating with respect to time any solution curve $(y(t), z(t))$ of (A.2) which belongs to the manifold in question, i.e., satisfies $y(t) = \pi(z(t))$. In other words, a center manifold for (A.2) can be described as the graph of a mapping $y = \pi(z)$ satisfying the partial differential equation (A.3b), with the constraints specified by (A.3a).

Note that the previous statement describes the *existence*, but not the *uniqueness* of a center manifold for (1.2): in fact, a system may have different center manifolds. Note also that if g and f are C^∞ functions, the system (A.2) has a C^k center manifold for any $k \geq 2$, but not necessarily a C^∞ center manifold.

Theorem: Suppose $y = \pi(z)$ is a center manifold for (A.2) at $(0, 0)$. Let $(y(t), z(t))$ be a solution curve of (A.2). There exists a neighborhood U° of $(0, 0)$ and real numbers $M > 0$ and $K > 0$ such that, if $(y(0), z(0)) \in U^\circ$, then

$$\|y(t) - \pi(z(t))\| \leq Me^{-Kt} \|y(0) - \pi(z(0))\|$$

for all $t \geq 0$, as long as $(y(t), z(t)) \in U^\circ$. ■

This theorem shows that any trajectory of the system (A.2) starting at a point sufficiently close to $(0, 0)$, i.e., close to the point at which the center manifold has been defined, converges, as time tends to infinity, to a trajectory which belongs to the center manifold.

The following statement shows to what extent center manifolds are useful in the analysis of the asymptotic properties of the system (A.2) near $(0, 0)$. Recall that, by definition, any trajectory of (A.2) starting at a point $y^\circ = \pi(z^\circ)$ of a center manifold can be described in the form

$$y(t) = \pi(\zeta(t)), \quad z(t) = \zeta(t)$$

with $\zeta(t)$ the solution of

$$\dot{\zeta} = B\zeta + f(\pi(\zeta), \zeta) \tag{A.4}$$

satisfying the initial condition $\zeta(0) = z^\circ$. The essence of the following result is that the asymptotic behavior of (A.2), for small initial conditions, is completely determined by its behavior for initial conditions *on the center manifold.*, i.e., by the asymptotic behavior of (A.4).

Theorem (Reduction Principle): Suppose $\zeta = 0$ is a stable (respectively, asymptotically stable, unstable) equilibrium of (A.4). Then $(y, z) = (0, 0)$ is a stable (respectively, asymptotically stable, unstable) equilibrium of (A.2).

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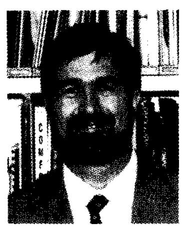

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