Department of Electrical & Computer Engineering
University of California, Santa Barbara

ECE 245
Fall 2009
Shynk
H.O. #21

EXAMPLE FINAL EXAM

INSTRUCTIONS

This exam is open book and open notes. It consists of 4 problems and is worth a total of 160 points. The problems are not of equal difficulty so use discretion in allocating your time. Attempt to answer all questions in any order.

1. WIENER FILTERING (40 points)

Let the signal $s(n)$ be transmitted across a fixed channel with the following transfer function:

$$H(z) = 1 + 2z^{-1}. \tag{1}$$

Assume that $s(n)$ is a zero-mean white random sequence with variance $\sigma_s^2 = 1$. The channel output $x(n)$ is processed by a filter (known as an equalizer) to compensate for $H(z)$.

(a) Determine the noncausal IIR Wiener filter that minimizes the mean-square error (MSE) between the equalizer output $y(n)$ and the transmitted signal $s(n)$. Specify the corresponding minimum MSE.

(b) Assume now that the equalizer is restricted to be a causal FIR filter with the following transfer function:

$$G(z) = a + bz^{-1}. \tag{2}$$

Repeat part (a) using $G(z)$ (i.e., find a, b, and the minimum MSE).

(c) Consider instead using a one-step linear predictor instead of $G(z)$ with transfer function

$$F(z) = c + dz^{-1}, \tag{3}$$

i.e., predicting $x(n)$ using $\{x(n-1), x(n-2)\}$. Find the linear predictor that minimize the MSE, and determine the corresponding minimum MSE. Discuss how these results differ from those obtained in part (b).

(d) Instead of the MSE, consider the following cost function:

$$\xi = E[(y^2(n) - 1)^2] \tag{4}$$

where $y(n)$ is the equalizer output. For the equalizer transfer function $G(z)$, determine the coefficients that minimize ξ. In order to proceed, assume that $s(n)$ is a Gaussian sequence (even though practical communication signals are not Gaussian).

2. LMS ALGORITHM (40 points)

The sign-input least-mean-square (LMS) algorithm is given by

$$W(n+1) = W(n) + 2\mu e(n)\mathrm{sgn}(X(n)) \tag{5}$$

where the weight vector $W(n)$ has N components, $X(n) = [x(n), \ldots, x(n-N+1)]^T$ is the input signal vector, and $\mu \geq 0$ is the step-size parameter. The signum function retains the sign of each component of $X(n)$. The error signal has the usual form $e(n) = d(n) - W^T(n)X(n)$ where $d(n)$ is the desired signal. Assume that $X(n)$ and $d(n)$ are jointly Gaussian with zero mean. For two jointly Gaussian random variables U and V with zero mean, it can be shown that

$$E[U\mathrm{sgn}(V)] = \frac{1}{\sigma_v}\sqrt{\frac{2}{\pi}}E[UV] \tag{6}$$

where $\sigma_v^2 = E[V^2]$.

(a) Find an expression for $E[\mathrm{sgn}(X(n))X^T(n)]$ in terms of $R = E[X(n)X^T(n)]$ and $\sigma_x^2 = E[x^2(n)]$.

(b) Find the converged weight vector W_o for the algorithm above (assuming that μ is chosen such that the algorithm converges).

(c) Define the weight-error vector $V(n) = W(n) - W_o$ and find a recursive expression for $V(n+1)$ in terms of $V(n)$, $X(n)$, μ, and $e_o(n) = d(n) - W_o^T X(n)$.

(d) For convergence in the mean, use your result in part (c) and the independence assumption to find a stability bound for μ.

3. LEAST-SQUARES FILTERING (40 points)

Consider the following scalar measurement equation:

$$d(i) = w + e_o(i) \tag{7}$$

for $i \geq 1$ where w is fixed (nonrandom) and $e_o(i)$ is white noise with zero mean and variance σ^2.

(a) Assuming there are n measurements $\{d(1), \ldots, d(n)\}$, find the least-squares (LS) estimator $\hat{w}(n)$ (assume that the forgetting factor is $\lambda = 1$).

(b) Show that $\hat{w}(n)$ is unbiased and find its variance.

(c) Using your result in part (a), find a recursive LS (RLS) estimator for w, i.e., find an expression for $\hat{w}(n)$ in terms of $\hat{w}(n-1)$. Demonstrate that this recursion converges in the mean using the translated coordinate weight $\hat{v}(n) = \hat{w}(n) - w$.

(d) Let the variance in part (b) be denoted by $r(n)$. Find an expression for $r(n)$ only in terms of $r(n-1)$ and σ^2 (by first finding a recursion for $r^{-1}(n)$).

(e) Repeat part (a) for the following model:

$$d(i) = wu(i) + e_o(i) \tag{8}$$

for $i \geq 1$ where $u(i)$ is a deterministic process.

4. LATTICE FILTERING (40 points)

Let the prediction errors of the mth lattice stage be given by

$$\begin{aligned} f_m(n) &= f_{m-1}(n) + \Gamma_{m,f}(n) b_{m-1}(n-1) \quad (9) \\ b_m(n) &= b_{m-1}(n-1) + \Gamma_{m,b}(n) f_{m-1}(n) \quad (10) \end{aligned}$$

where the reflection coefficients $\{\Gamma_{m,f}(n), \Gamma_{m,b}(n)\}$ are distinct.

(a) The input signal $f_0(n) = b_0(n) = u(n)$ is stationary with zero mean. By minimizing $E[f_m^2(n)]$, find an expression for the optimal weight vector $\Gamma_{m,f,o}$ in terms of the mean-square prediction errors of the previous stage. Similarly, find an expression for $\Gamma_{m,b,o}$ by minimizing $E[b_m^2(n)]$.

(b) Derive least-mean-square (LMS) adaptive algorithms to update the coefficients. Let the corresponding step-size parameters be denoted by $\{\mu_{m,f}, \mu_{m,b}\}$.

(c) Assume that the first $m-1$ stages have converged so that the corresponding prediction errors can be regarded as stationary signals. Determine the stability bounds for $\{\mu_{m,f}, \mu_{m,b}\}$ for convergence in the mean. Also, specify the stability bounds if the reflection coefficients of the first $m-1$ stages are all zero (as they would be when the system is initialized). Do you think the stability bounds for the mth stage become tighter as the first $m-1$ stages adapt, i.e., as the reflection coefficients adapt from zero to their Wiener values?

(d) Consider the joint process estimator with coefficients $\{h_m(n)\}$. Derive a gradient adaptive algorithm to update these coefficients. Let the step-size parameters be denoted by $\{\mu_m\}$.

(e) Assume the lattice predictor has converged. Find the stability bound for μ_k for convergence in the mean, and specify which lattice stage has the greatest influence on this bound.