LECTURE #8

– Capacity of fading channels:

System model:

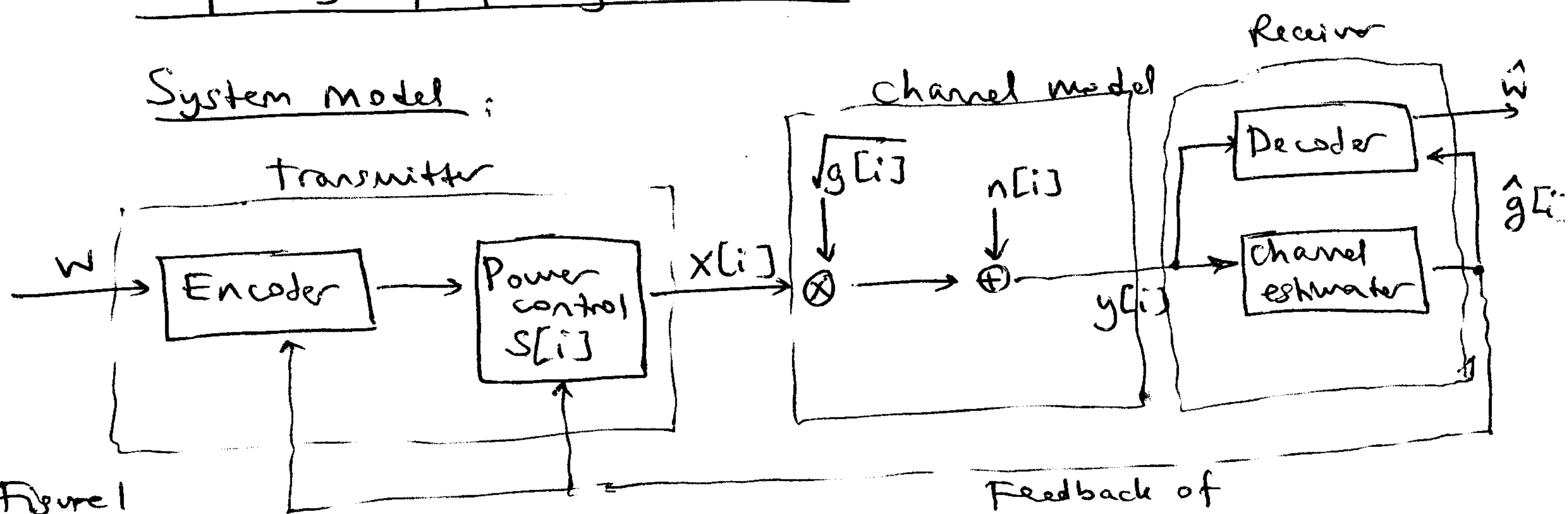


Figure 1

AWGN channel: [g[i] = constant in above system model]

- Let γ denote the received SNR. i: symbol index.
- For AWGN channel, $C = B \log_2(1+\gamma)$

[• B denotes the bandwidth in Hz.]

Fading channels: (g[i] variable). ~~from symbol to~~

Different cases,

(A) No side information: [nothing is known about channel fading].

It turns out that the only way to guarantee that the error probability (Pe) is arbitrarily small, is to assume worst-case fading.

Let $\gamma_{min} \triangleq \min_{\gamma : p(\gamma) > 0} \gamma$ [i.e. γ_{min} is the minimum SNR over the set of SNR's with non-zero probability.]

Then:

$$\boxed{C = B \log_2(1+\gamma_{min})}$$ (bps) (no side information)

(c) Side information at both the transmitter and receiver:

— Assume that $g[i]$ is known at both the transmitter and the receivers

Wolfowitz's result:

Let $c[i]$ be a stationary, ergodic stochastic process that represents the channel state.

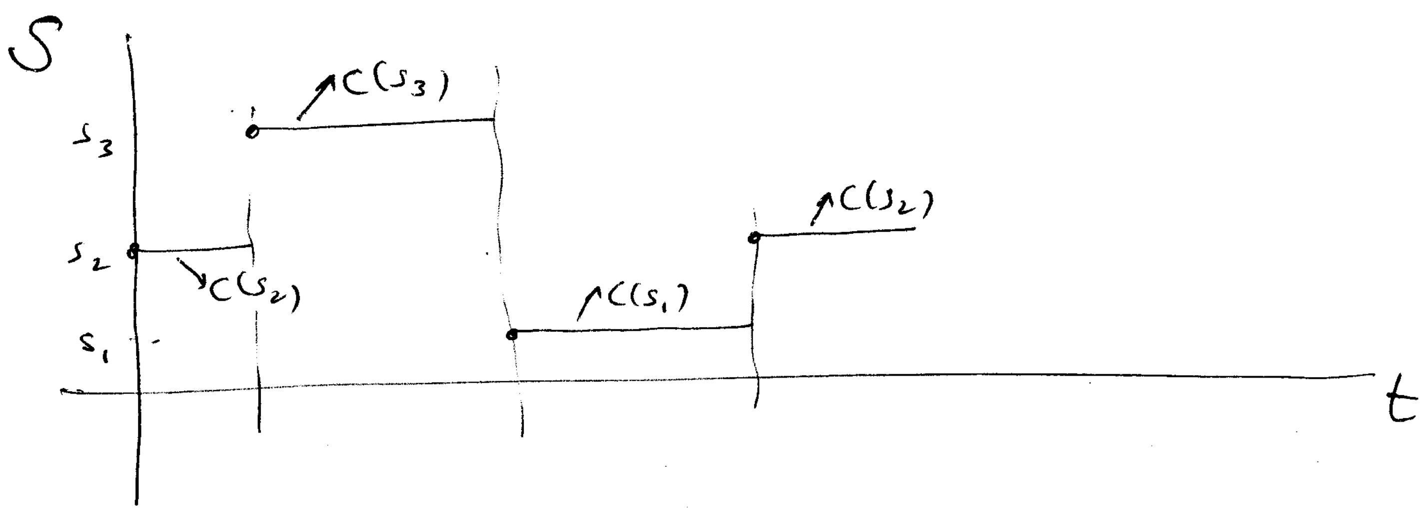
Assume that $c[i]$ takes values on a finite set S of discrete memoryless channels (DMC's) [A DMC is characterized by a ~~~~ probability transition matrix: $p(y|x)$]

- Let C_s denote the capacity of a particular channel $s \in S$

and let $p(s)$ denote the probability (or fraction of time) that the channel is in state s

- Wolfowitz showed that the capacity of this time-varying channel is:

$$\boxed{C = \sum_{s \in S} C_s \, p(s)}$$



- Now go back to the channel model in Figure 1.
- Assume that channel gain $g[i]$ is stationary and ergodic.

- If the $g[i]$ is known both at the transmitter & the receiver, then $g[i]$ becomes deterministic (for each i).

  Then, Wolfowitz's result gives:
  (extended to continuous SNR range.)

$$\boxed{C = \int_0^\infty C(\gamma) \cdot p(\gamma)\, d\gamma} \quad \text{[no power adaptation, only Rate adaptation]}$$

$$= \boxed{\int_0^\infty [B \log_2(1+\gamma)] \cdot p(\gamma)\, d\gamma} \quad \Big(= E_\gamma [\underbrace{B \log_2(1+\gamma)}_{\text{concave function}}]\Big)$$

- Note that by Jensen's inequality
  [which says that $E[f(X)] \leq f(E(X))$ f concave.]

$$\le B \cdot \log_2 \left(E[1+\gamma]\right)$$

$$= \boxed{B \cdot \log_2 (1+\bar{\gamma})}$$

- This shows that the capacity of the fading channel (under transmitter and receiver side information) is <u>less than or equal to</u> the AWGN channel with the <u>same average SNR</u>.

* Why does this hold? Because the above scheme assumes <u>no</u> power adaptation (and uses only rate adaptation), <u>i.e.</u> The <u>implicit assumption</u> above is that the TX power

is constant. [This is a drawback of formulating capacity with only receiver metrics.]

→ If the channel is fading, the right thing to do is to vary the Tx power accordingly.

- Let $S(\gamma)$ denote the transmit power adjusted with respect to changes in SNR $(=\gamma)$, as reported by the receiver in the feedback loop.

- However, we still need to have an average power constraint at the transmitter:

$$\int_0^\infty S(\gamma)\, p(\gamma)\, d\gamma \leq S.$$

denotes the maximum allowed average transmit power.

Thm: ~~Definition~~ The channel capacity of the fading channel, with power & rate adaptation (at transmitter), and subject to an average transmit power constraint S, is given by:

$$C(S) = \max_{S(\gamma):\ \int S(\gamma)p(\gamma)\,d\gamma \leq S} \int_0^\infty B \log_2\left(1 + \frac{S(\gamma)\,\gamma}{S}\right) p(\gamma)\, d\gamma.$$

i.e. $\exists$ a channel coding scheme with average power S that achieves $C(S)$ with arb. small P_e.

Converse: Any coding scheme with $R > C(S)$ and avg power S cannot achieve arb. low P_e.

Solution to maximization problem:

(use Lagrange multipliers):

$$\frac{S(\gamma)}{S} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

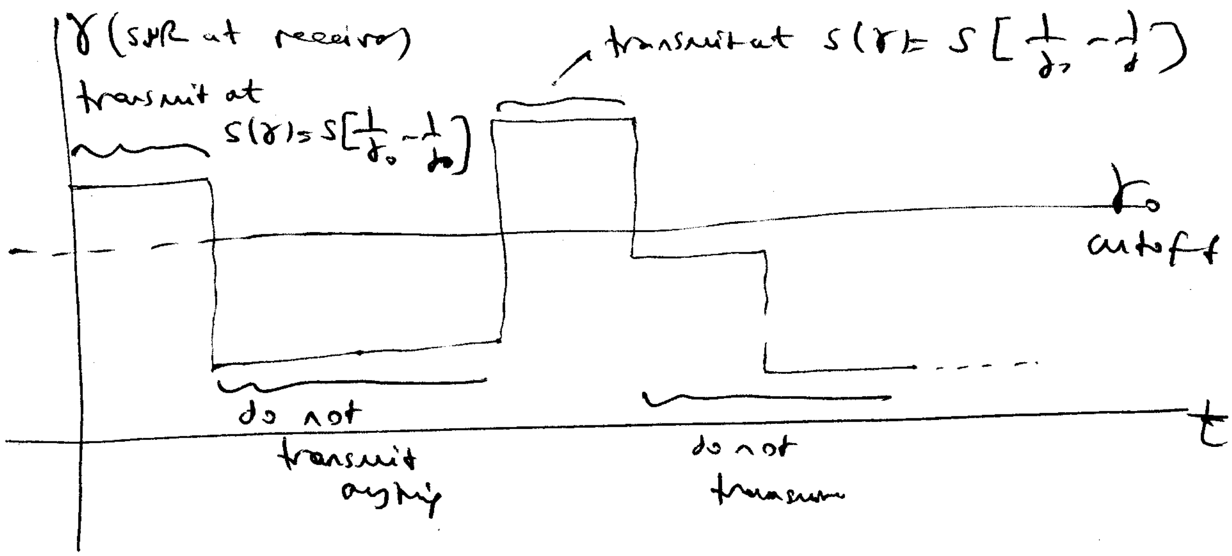
That is, if the SNR ($=\gamma$) is below some "cut-off" value γ_0, then do not transmit anything. If $\gamma > \gamma_0$, then use $S(\gamma) = S \cdot \left[\frac{1}{\gamma_0} - \frac{1}{\gamma}\right]$ amount of power (under those channel conditions.)

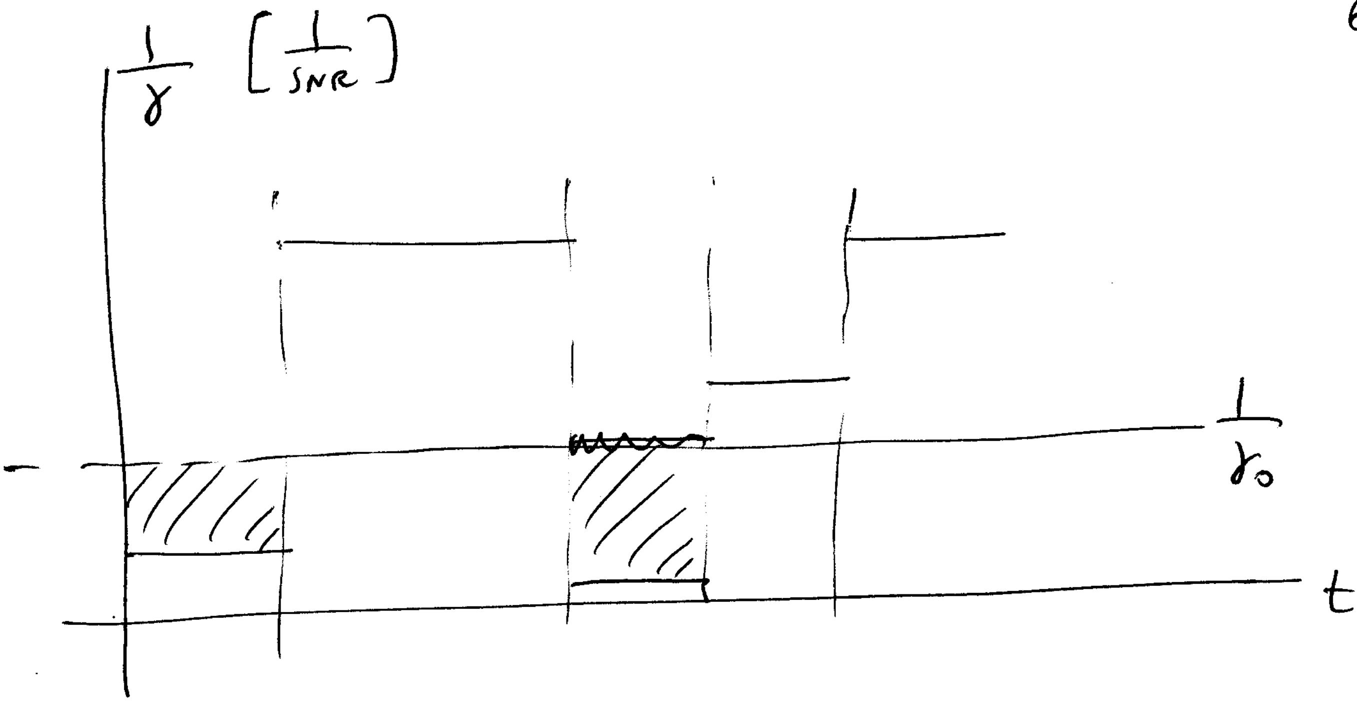
- γ_0 is computed by: solving:

$$\int_{\gamma_0}^{\infty} \left[\frac{1}{\gamma_0} - \frac{1}{\gamma}\right] p(\gamma)\, d\gamma = 1.$$

— called "water-pouring in time".

Intuition:





– We pour energy (in time) to this structure [we pour it like water.] – the energy goes to where $\frac{1}{\gamma}$ is low.

- Note that at high S, all the γ's may be used.
  At low S, only a few γ's will be used.

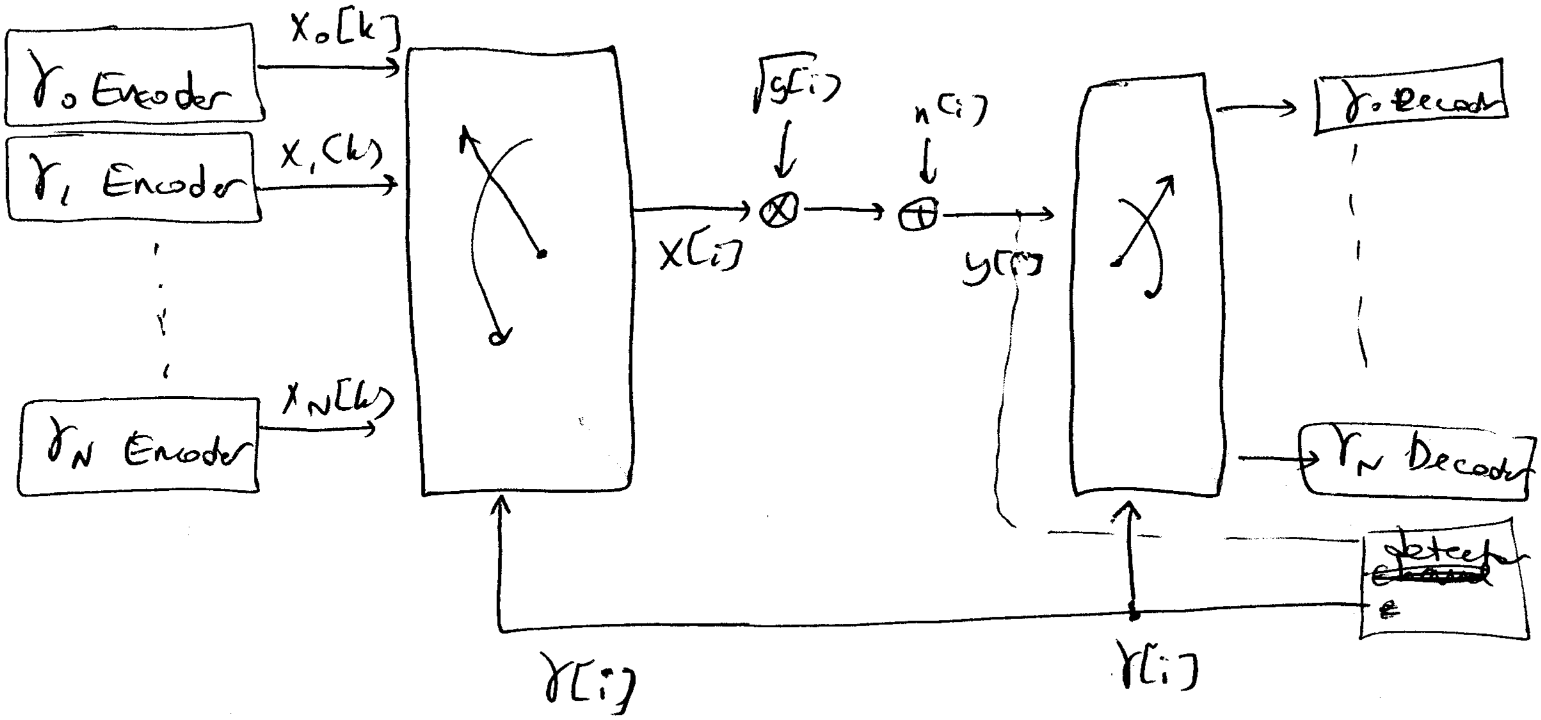
With this, the closed form capacity formula for power + rate adaptation under tx + rx side info is:

$$\boxed{C(S) = \int_{\gamma_0}^{\infty} B \log_2\left(\frac{\gamma}{\gamma_0}\right) p(\gamma)\, d\gamma}$$

How do you achieve this capacity?

– <u>Main idea</u>: time diversity with multiplexed input & output.

- Quantize the γ's into $\{\gamma_j : 1 \le j \le N\}$.
- For each γ_j, design an encoder/decoder pair with average power $S(\gamma_j)$ and $R_j \approx C_j$ where C_j = capacity of a time-invariant AWGN channel with received SNR $\frac{S(\gamma_j)}{S} \cdot \gamma_j$.



(c) Side info. at receiver; no side info at transmitter.

- If $g[i]$ is known at the receiver, then becomes [for that symbol i], deterministic path. $\Rightarrow$ AWGN channel with noise power $\frac{N_0 B}{g[i]}$. [Note that $\exists$ no power adaptation & no rate adaptation in this case since $g(i)$ not ~~known~~ known at transmitter.]
- Assume TX power fixed at S.
- If $\underline{g[i] \text{ is i.i.d.}}$, then the optimal input is an i.i.d. Gsn with avg power S.

$$\boxed{C(S) = \int_0^\infty B \log_2(1+\gamma)\, p(\gamma)\, d\gamma}$$

- no ~~tx~~ side info at T [hence no power + no rate adaptation]
- $g[i]$ iid.

This is the same capacity as [side info at tx + receiver, with rate adaptation but no power adaptation].

- Implies that: Without power adaptation, transmitter side information does not increase capacity under iid. fading $g[i]$

~~[However, tx side information with rate adaptation may decrease the complexity of the receiver.]~~

[Question: why is that? why doesn't Rate only adaptation help capacity?]

* Note that the above result holds for iid fading $g[i]$.
Real channels ~~are~~ have correlated fading.

~~• If the fading is corre~~

– When you have correlated fading:

- Code design ~~at~~ must incorporate channel correlation statistics. (Tx)
- Rx: MLD's complexity ~ channel decorrelation time.

∴ Correlated fading channels: Adaptive transmission has ~~tot~~ lower complexity (and higher capacity)

Channel inversion.

– Qualcomm's IS-95 ~~[illegible]~~ system used channel inversion at the receiver. The main idea was to ~~[illegible]~~ receive each user's transmission at a constant ~~with an equal power level~~ SNR. – The receiver tries to invert the fading to make it look like an AWGN channel.

· This is a suboptimal scheme for fading channels

Under channel inversion: $\frac{S(\gamma)}{S} = \frac{\sigma}{\gamma}$ $\left[\sigma = \frac{S(\gamma)}{S}\cdot\gamma\right]$

where σ = constant SNR to be maintained under the Tx power constraint..

Then: $\cancel{\int_0^\infty \frac{S(\gamma)}{S}\gamma\, p(\gamma)\, d\gamma = 1 \Rightarrow \int_0}$

$$\int_0^\infty \frac{\sigma}{\gamma} p(\gamma)\, d\gamma = 1. \quad \left[\text{since } \int \underbrace{\frac{S(\gamma)}{S}}_{\frac{\sigma}{\gamma}} p(\gamma) = 1\right]$$

$$\Leftrightarrow \sigma = \frac{1}{\int_0^\infty \frac{1}{\gamma} p(\gamma)\, d\gamma} = \frac{1}{E_\gamma\left[\frac{1}{\gamma}\right]}$$

Under channel inversion:

$$\boxed{C(S) = B\log_2(1+\sigma) = B\log_2\left[1 + \frac{1}{E[1/\gamma]}\right]}$$

- Rayleigh fading: $E\left[\frac{1}{\gamma}\right] = +\infty$, hence $C(S) = 0$ under Rayleigh fading.

Truncated inversion policy:

$$\frac{S(\gamma)}{S} = \begin{cases} \frac{\sigma}{\gamma}, & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

Then: $$\int_{\gamma_0}^{\infty} \frac{\sigma}{\gamma} p(\gamma)\, d\gamma = 1$$

$$\Rightarrow \sigma = \frac{1}{\int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma)\, d\gamma}$$

- The receiver must know when $\gamma < \gamma_0$.

$$\boxed{C(S) = \max_{0 < \gamma_0 < +\infty} B \cdot \log_2 \left[1 + \underbrace{\frac{1}{\int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma)\, d\gamma}}_{\sigma} \right] p(\gamma \geq \gamma_0)}$$

OR can set γ_0 to obtain $P_{out} = P_r(\gamma < \gamma_0)$

Adaptive Modulation:

– Variable-Rate Variable-Power MQAM:

- Fix symbol rate T_s.
- M-QAM (M = # points in the constellation)
- Assume ideal Nyquist pulses $\mathrm{sinc}\left(\frac{t}{T_s}\right)$.

Definitions:

$\bar{S} \triangleq$ avg power constraint at transmitter.

$$\int_0^\infty S(\gamma)\, p(\gamma) \le \bar{S}.$$

$B \triangleq$ bandwidth (in Hz).

$N_0 \triangleq$ noise spectral density (W/Hz)

$$\gamma[i] = \frac{\bar{S} \cdot g[i]}{N_0 \cdot B} \qquad \text{(SNR of the ith symbol)}.$$

$$\bar{\gamma} = \frac{\bar{S}}{N_0 B}$$ ~~[It is assumed that E[g[i]] = 1 and g[i] are independent]~~

average SNR

$$\frac{\bar{E_s}}{N_0} = \frac{\bar{S} \cdot T_s}{N_0} = \bar{\gamma}$$

$\frac{\text{avg per symbol}}{\text{noise spectral density}}$

- For fixed M,

$$R = \frac{\lfloor \log_2 M \rfloor}{T_s} \quad \text{(data rate in bps.)}$$

M-QAM modulation:

(with ideal coherent phase detection)

$$BER \leq 2\, e^{-1.5\bar{\gamma}/(M-1)} \quad \text{(AWGN channel)} \atop \text{M-QAM}$$

For $M \geq 4$, and $0 \leq \bar{\gamma} \leq 30\,dB$, a tighter bound:

$$BER \leq 0.2\, e^{-1.5\bar{\gamma}/(M-1)} \quad \text{(AWGN channel)} \atop \text{M-QAM}$$

- Last time (lecture 7), we showed that $\overline{BER} = \int_0^\infty BER(\gamma) \cdot p(\gamma)\, d\gamma$ assuming no power or rate adaptation.

[Recall that for Rayleigh fading: $BER = \frac{1}{4\bar{\gamma}}$ at large SNR's
+ BPSK $(M=2)$
without adaptation: $\overline{BER} = 10^{-3}$ requires $\bar{\gamma} = 24\,dB$.

For $M \geq 4$: $\overline{BER} \leq \int_0^\infty 0.2\, e^{-1.5\gamma/(M-1)}\, p(\gamma)\, d\gamma$, $M \geq 4$

$\overline{BER} = 10^{-3} \Rightarrow$ SNR: $\bar{\gamma} = 26\,dB$ for $p = 2$.]

Now, under adaptive modulation:

$$\mathrm{BER}(\gamma) \le 0.2\, e^{-1.5\left[\gamma \cdot \frac{S(\gamma)}{\bar{S}}\right]/(M-1)}$$

Idea: Adjust M and $S(\gamma)$ to maintain a fixed BER

$$\Leftrightarrow \quad M(\gamma) = 1 + \frac{1.5\,\gamma}{-\ln(5\,\mathrm{BER})} \frac{S(\gamma)}{\bar{S}}.$$

target BER

The spectral efficiency for M-QAM modulation is $\log_2 M$.

$$\max E[\log_2 M(\gamma)] = \int_0^\infty \log_2\left(1 + \frac{1.5\,\gamma}{-\ln(5\,\mathrm{BER})} \frac{S(\gamma)}{\bar{S}}\right) p(\gamma)\,d\gamma$$

$$\text{s.t.} \quad \int S(\gamma)\,p(\gamma)\,d\gamma \le \bar{S}.$$

Solution:

$$\frac{S(\gamma)}{\bar{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_k} & \gamma \ge \boxed{\frac{\gamma_0}{K}} \equiv \gamma_k \\ 0 & \gamma < \boxed{\frac{\gamma_0}{K}} \equiv \gamma_k \end{cases}$$

where $K \equiv \dfrac{-1.5}{\ln(5\,\mathrm{BER})}$

Define: $\gamma_k \equiv \dfrac{\gamma_0}{K}$

Then:

$$\boxed{\max E[\log_2 M] = \frac{R}{B} = \int_{\gamma_k}^{\infty} \log_2\left(\frac{\gamma}{\gamma_k}\right) p(\gamma)\, d\gamma}$$

shows that
*∴ Uncoded M-QAM has an effective power loss of K, relative to the optimal transmission scheme (that uses continuous power levels), independent of fading distribution.

[i.e. the distribution of g[i]'s does not matter.]

Hence: K is also the maximum possible coding gain.

Channel inversion (using M-QAM):

$$\frac{R}{B} = \log_2\left(\boxed{1 + \boxed{\frac{-1.5}{\ln(5BER)\, E[1/\gamma]}}}\right) \quad \text{for } M \geq 4, \; 0 \leq \bar{\gamma} \leq 30\text{dB}$$

(since $\frac{S(\gamma)}{S} = \frac{\sigma}{\gamma}$ for channel inversion)

$[\log_2 M = \log_2(1 + K\cdot\sigma)]$.

$$\sigma = \frac{1}{E[1/\gamma]}.$$

see next page

$$\frac{S(\gamma)}{\bar{S}} = \frac{\sigma}{\gamma}.$$

$$M(\gamma) = 1 + \frac{1.5}{-\ln(5BER)} \cdot \boxed{\frac{\overbrace{\gamma\, S(\gamma)}^{\sigma}}{\bar{S}}}$$

$$\frac{R}{B} = \log_2 M(\gamma) = \log_2\left[1 + \frac{1.5}{-\ln(5BER)} \cdot \sigma\right]$$

$\uparrow$

for M-QAM with channel inversion

$$\boxed{= \log_2\left[1 + \frac{-1.5}{\ln(5BER) \cdot E[1/\gamma]}\right]}$$

Truncated channel inversion w/ M-QAM:

$$\frac{R}{B} = \max_{\gamma_0} \log_2\left[1 + \frac{-1.5}{\ln(5BER)\, E_{\gamma_0}[1/\gamma]}\right] p(\gamma > \gamma_0).$$

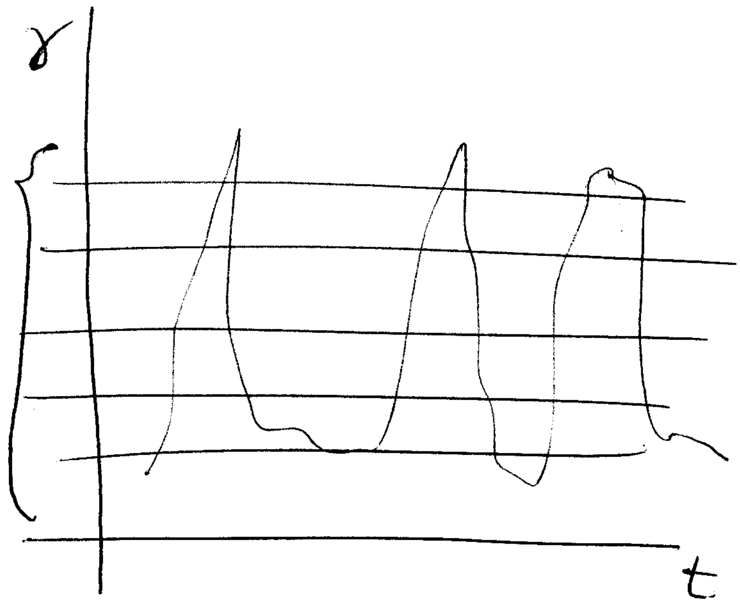
$$\int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma)\, d\gamma$$

Constellation restriction:

$M \in \{2, 4, 16, 64, \ldots\}$

Square constellations.

discretize to N+1 fading regions



Associate M_j with jth fading region.

Region boundaries.

$$M(\gamma) = \frac{\gamma}{\gamma_k^*}$$

where $\gamma_k^* > 0$ is a parameter to be optimized.

Define $M_{N+1} = \infty$

For fixed γ, transmit the largest constellation possible using M_j's from the set $\{2, 4, 16, 64, 256\}$

EX $2 \le \frac{\gamma}{\gamma_k^*} < 4 \Rightarrow$ use 2-QAM

Region boundaries: $\boxed{\gamma = \gamma_k^* \cdot M_j}$

- The above shows how to pick the $M(\gamma)$ using M_j's.

Power control:

- satisfy a target BER ($\forall M_j > 0$)

$$\frac{S_j(\gamma)}{\bar{S}} = \begin{cases} (M_j - 1)\frac{1}{K} & M_j \le \frac{\gamma}{\gamma_k^*} \le M_{j+1} \\ 0 & M_j = 0. \end{cases}$$

Fixed BER:

$$\frac{E_s(j)}{N_0} = \frac{\gamma S_j(\gamma)}{\bar{S}} = \frac{M_j - 1}{K}$$

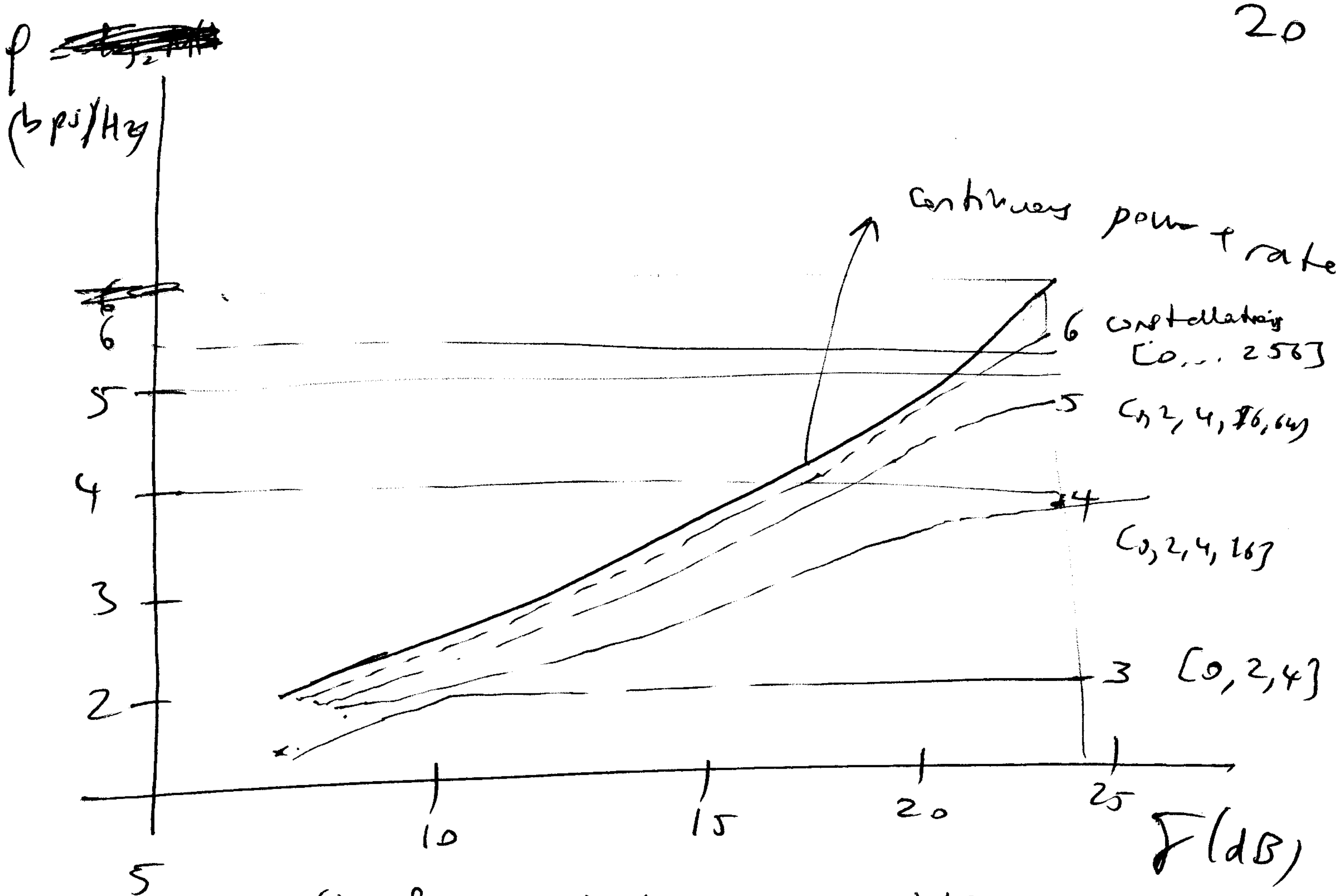
Region (j)	γ	M_j	$S_j(\gamma)/\bar{S}$ $\left(\uparrow \frac{E_s(j)}{N_o}\right)$
0	$0 \le \frac{\gamma}{\gamma_k^*} < 2$	0	0
1	$2 \le \frac{\gamma}{\gamma_k^*} < 4$	2	$\frac{1}{K \cdot \gamma}$
2	$4 \le \frac{\gamma}{\gamma_k^*} < 16$	4	$\frac{3}{K \cdot \gamma}$
3	$16 \le \frac{\gamma}{\gamma_k^*} < 64$	16	$\frac{15}{K \cdot \gamma}$
4	$64 \le \frac{\gamma}{\gamma_k^*} < +\infty$	14	$\frac{63}{K \cdot \gamma}$

$$\frac{R}{B} = \sum_{j=1}^{N} \log_2 (M_j) \cdot p\left(M_j \le \gamma/\gamma_k^* < M_{j+1} \right).$$

Finally, ~~find~~ maximize this w.r.t. γ_k^*.

$$\sum_{j=1}^{N} \int_{\gamma_k^* M_j}^{\gamma_k^* M_{j+1}} \frac{S_j(\gamma)}{\bar{S}} p(\gamma)\, d\gamma = 1.$$

(Use numerical search — no closed form)

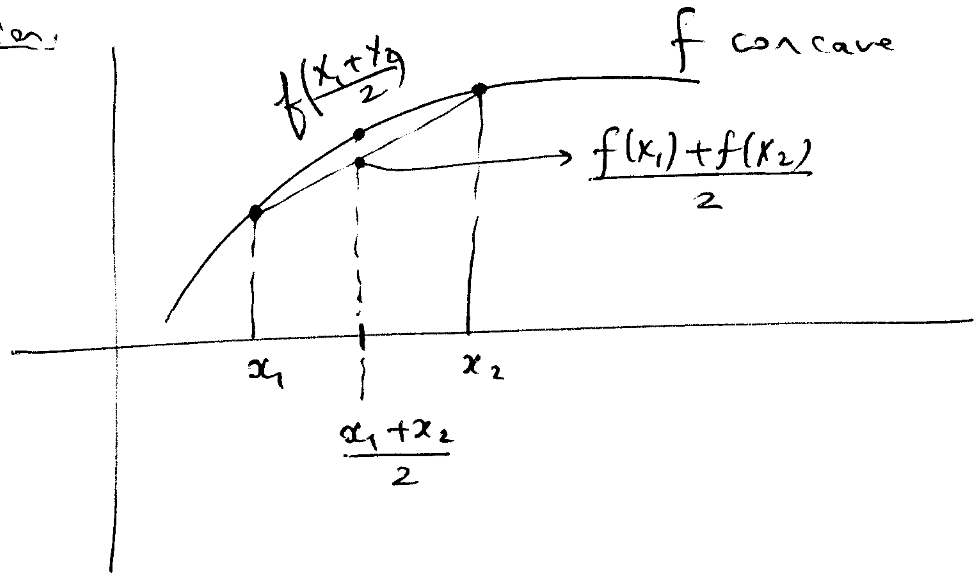


(in lognormal shadowing σ = 8 dB)

Jensen's Inequality

$$E[f(X)] \leq f(E(X)) \qquad f \text{ concave.}$$

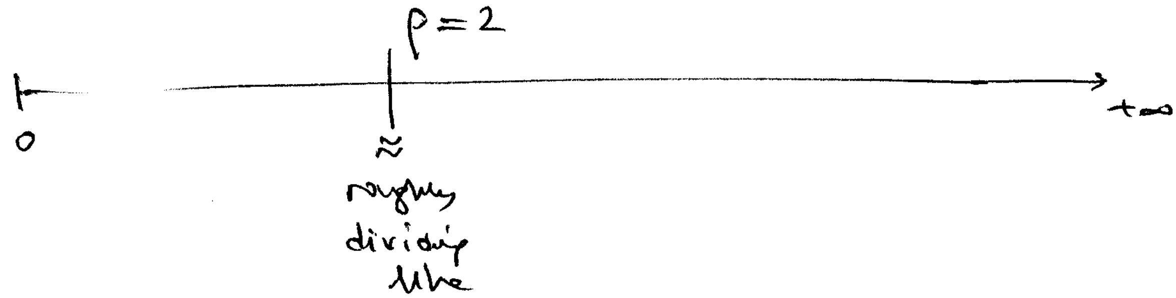
Intuition:



[f overestimates]

LECTURE #7 [Appendix]

1) Power-limited vs. bandwidth limited regimes:



Relationship to SNR:

- For a fixed E_b/N_0, since $\frac{E_b}{N_0} = \frac{SNR}{\rho}$,

  as $\rho \downarrow$, $SNR \downarrow$. So for fixed E_b/N_0, small ρ's correspond to small SNR's