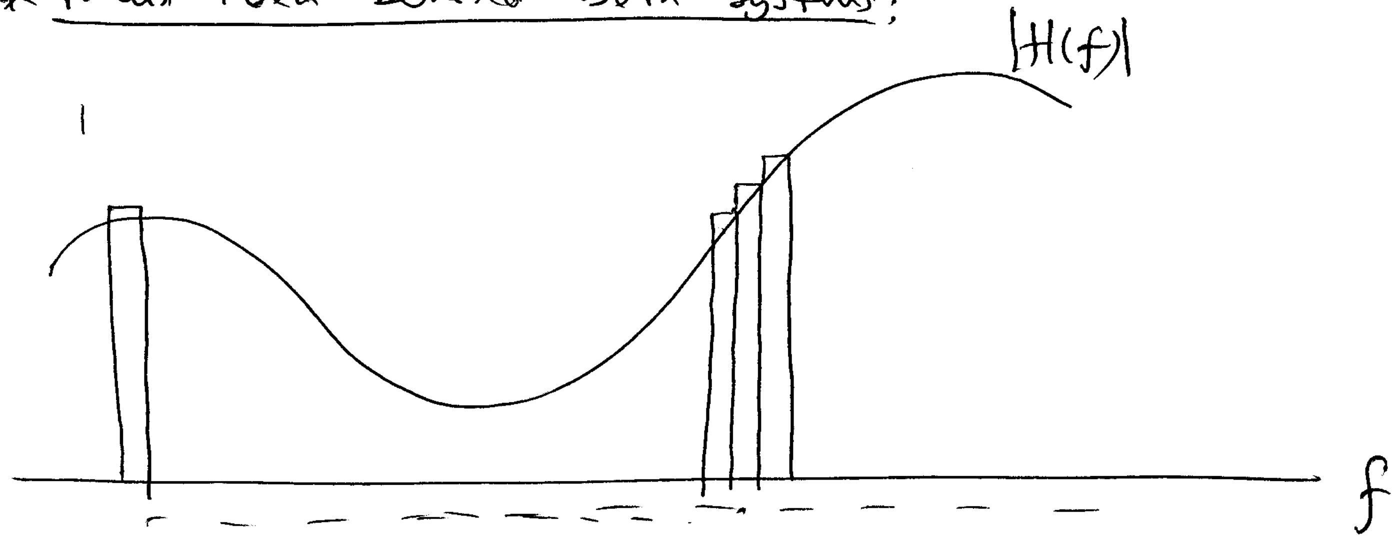
LECTURE - DMT & OFDM

- main advantage of OFDM on wireless channels: reduction (or elimination) of ISI without the use of equalization.
  - DMT: used in phone channels (where channel is nearly static compared to wireless channels)
  - OFDM: wireless systems where channel gain may change or unknown.

* main idea behind both systems:



- use many "subcarriers" next to each other in the frequency domain.

- If the bandwidth of a subcarrier is small enough (more specifically $<< B_c$ = coherence bandwidth), then that subcarrier will see a *flat* channel.

[illegible]

- OFDM: places the same amount of energy on each subcarrier:
  - creates a diversity technique: if some freqs are fading out, others may be in good shape.
  - no equalization needed.
  - [FH-SS: is a system where you have subcarriers; and the system (transmitter) hops around in a pattern of frequencies known to the receiver.]
- DMT: Discrete multitone modulation

  ~~places~~ allocates the energy per symbol optimally between the subcarriers.

  (# of subcarriers fixed: e.g. N = 16, 64, 128, 256, 512, 1024)

* Important info theory result: As N → ∞, the DMT system → the Shannon capacity of the ISI channel. (waterfilling argument)

Ⓐ VECTOR CODING

~~Mathematics:~~

~~VECTOR CODING~~

Ⓘ. Vector channel description of the ISI channel.

$$\boxed{H(D)\ \cancel{h[n]} = p_0 + p_1 D + p_2 D^2 + \dots + p_v D^v.}$$

$$\left(h[n] = p_0\,\delta[n] + p_1\,\delta[n-1] + \dots + p_v\,\delta[n-v] \right).$$

v : is called the ~~constant~~ "channel length" or # of taps in the channel

is:

$$\begin{bmatrix} Y_{N-1} \\ Y_{N-2} \\ \vdots \\ \vdots \\ Y_0 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & \cdots & p_v & \overbrace{0 \;\cdots\; 0}^{v \text{ extra entries here}} \\ 0 & p_0 & \cdots & \cdots & p_v \;\; 0 \cdots 0 \\ & & \ddots & & \\ & & & & \\ & & & p_0 & \cdots \;\; p_v \end{bmatrix} \begin{bmatrix} X_{N-1} \\ X_{N-2} \\ \vdots \\ X_0 \\ X_{-1} \\ \vdots \\ X_{-v} \end{bmatrix} + \begin{bmatrix} \eta_{N-1} \\ \vdots \\ \vdots \\ \vdots \\ \eta_0 \end{bmatrix}$$

$(N \times 1)$ $(N \times (N+v))$ $(N+v) \times 1$ $(N+v) \times$

Sizes:

$\uparrow$ A "Toeplitz" matrix

$$\boxed{\underset{\sim}{y} = P\,\underset{\sim}{x} + \underset{\sim}{n}.}$$

$P \downarrow$ $N \times (N+v)$

Vector ISI channel model

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Ⅱ — Now take the SVD of P.

$$\cancel{\underset{N\times(N+v)}{P} = \underset{N\times N}{F}\cdot\Big[\underset{\substack{N\times N\\ \text{diagonal}}}{\Lambda}\Big]\cdot M^{*}}$$

Hermitian..

$$P = \underset{N\times N}{\underbrace{F}}\cdot\Big[\underset{\substack{N\times N\\ \text{diagonal}}}{\Lambda}\;\Big|\;\underset{N\times v}{0}\Big]\,\underset{(N+v)\times(N+v)}{M^{*}}$$

Properties ; $FF^{*} = F^{*}F = I_{(N)}$ (F is unitary)

$MM^{*} = M^{*}M = I_{(N+v)}$ (M is unitary)

$\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$

λ_i : singular values. always: $\boxed{\lambda_i \geq 0}$

Ⅲ Vector coding:

Let $m_i \equiv$ first N columns of M.

f_i : " " " " F.

$$\underset{\substack{\text{modulated}\\ \text{~~Sign~~}\\ \text{signal}}}{\tilde{x}} = \sum_{n=0}^{N-1} \underset{\text{symbols}}{X_n}\cdot \underset{\substack{\text{distinct}\\ \text{modulation}\\ \text{functions}}}{m_n} = \underset{N\times N}{\tilde{M}}\,\underset{N\times 1}{\tilde{X}}. = \underset{(N+v)\times(N+v)}{M}\,\underset{(N+v)\times 1}{X}.$$

$$X = \begin{bmatrix} X_{N-1} \\ X_{N-2} \\ \vdots \\ X_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \tilde{X} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} v \end{matrix}$$

a): $$\underset{\sim}{y} = \underbrace{F\,[\Lambda \mid 0]\; M^*}_{\text{channel}}\; \overbrace{[\,M\underline{x}\,]}^{\text{the transmitted symbol}} + \underset{\sim}{n}.$$

↓ received signal

the modulation functions are picked this way for a reason

$$\overbrace{M^* M}^{I}\,\underline{x} = \underline{x}$$

$$= F\,[\Lambda \mid 0]\,\underline{x} + \underset{\sim}{n}.$$

$$= F\Lambda\underset{\sim}{x} + \underset{\sim}{n}.$$

b): $$Y \triangleq F^*\underset{\sim}{y} = F^*[F\Lambda\underset{\sim}{x} + \underset{\sim}{n}]$$

$$= \underbrace{F^* F}_{I}\,\Lambda\underset{\sim}{x} + F^*\underset{\sim}{n}$$

$$= \Lambda\underset{\sim}{x} + \underset{\sim}{N} \quad \text{where} \quad \underset{\sim}{N} \equiv F^*\underset{\sim}{n}.$$

$$\Rightarrow \boxed{Y_n = \lambda_n X_n + N_n.}$$ is the "channel partition".

∴ a set of independent, parallel ~~&~~ channels have been created.

$$\boxed{SNR_n = \lambda_n^2 \cdot \frac{\mathcal{E}_n}{N_0/2}}$$

~~III~~ • Autocorrelation of $\underset{\sim}{N}$:

$$\underset{\sim}{N} = F^* \underset{\sim}{n}.$$

$$R_N = F^* \underbrace{\boxed{R_{\underset{\sim}{n}}}}_{\downarrow} F = \frac{N_0}{2} \cdot F^* F = \frac{N_0}{2}.$$

diagonal: $\frac{N_0}{2} \cdot I$
since
$\underset{\sim}{n}$ is white.

~~Perform~~

* This scheme is optimal. ~~(See Cover; Ch 8); water filling~~

(IV) Performance: $\left(\boxed{\frac{1}{2}\log_2(1+SNR)} \right.$ shannon AWGN$\left.\right)$)

$$\bar b = \frac{1}{2\underbrace{\boxed{(N+\nu)}}_{\downarrow}} \sum_{n=0}^{N-1} \log_2\left(1 + \frac{\boxed{SNR_n}}{\boxed{\Gamma}}\right)$$

→ SNR of nth [illegible] channel

$(N+\nu)$ dimensions

↑ gap of modulation + coding scheme used in each subcarrier

[Note that the overhead is:

$\frac{\nu}{N+\nu}$. $\Rightarrow$ optimal as $N \to \infty$ for fixed ν.]

$$\triangleq \frac{1}{2}\log_2\left(1 + \frac{\boxed{SNR_{vc}}}{\Gamma}\right)$$

→ definition of $\boxed{\text{vector coding}}$ $\widehat{\ }$ SNR

$$SNR_{vc} \triangleq \Gamma \cdot \left[\left\{ \prod_{n=0}^{N-1} \left(1+\frac{SNR_n}{\Gamma}\right) \right\}^{1/\boxed{(N+v)}} - 1 \right]$$

overhead shows up here

≜ the equivalent SNR if single carrier transmission was used with the same gap

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Example: (vector coding):

$H(D) = 1 + \underbrace{0.9 D^{-1}}$. $\quad SNR_{MFB} = 10\,dB \quad \Gamma = 0\,dB$

→ a non-causal channel

$\Rightarrow v = 1 \qquad \boxed{\sigma^2 = 0.181}$

use $\underline{N=8}$.

$\rightarrow N+v = 9$.

①: $$P = \begin{bmatrix} 0.9 & 1 & & & \\ & 0.9 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0.9 & 1 \end{bmatrix}$$

8 x 9 Toeplitz matrix

② SVD in MATLAB:

$$\lambda = [1.871 \quad 1.786 \quad 1.646 \quad 1.407 \quad 1.224 \quad 0.954 \quad 0.657 \quad 0.344].$$

$$g_n = \frac{\lambda_n^2}{\sigma^2} \qquad \sigma^2 = 0.181$$

$$= [19.44 \quad 17.62 \quad 14.97 \quad 11.73 \quad 8.27 \quad 5.03 \quad 2.38 \quad 0.66].$$

③ Do waterfilling on those 11 channels: $\boxed{\cancel{\mathcal{E} = 9}}$ → energy per symbol

~~You'll see that the last channel not used. ie~~

$$\boxed{\cancel{K = \frac{1}{7}\left[9 + \sum_{n=0}^{7} \frac{1}{g_n}\right]}}$$

↑ waterfilling constant

$\mathcal{E} = 9$

\# ~~$\mathcal{E}$~~.

$$\mathcal{E}_x = \underbrace{\overline{\mathcal{E}}}^{\text{take } = 1.} \cdot \underbrace{(N+\nu)}_{9}.$$

↑ energy of symbol x

$\overline{\mathcal{E}}$: energy per dimension.

$$\boxed{\mathcal{E}_n = K - \frac{\Gamma}{g_n}}$$

K = 1.429:

n	g_n	$\mathcal{E}_n$	SNR_n
0	19.44	1.377	26.64
1			24.17
2			20.39
3			15.75
4			10.82
5			6.18
6			2.4

$$SNR_{vc} = \left[\prod_{n=0}^{6} (1+SNR_n) \right]^{1/9} - 1$$

$$= \underline{8.1\,dB}$$

Bitrate: $\bar{b} = \underline{\underline{1.45}}$ bits/dimension

$N \to \infty$: $SNR \to 8.8\,dB$

$\bar{b} \to \bar{c} = \underline{\underline{1.55}}$ bits/dimension

$\frac{1}{N} \simeq \frac{1}{N+\nu}$ N large.

Ⓑ **DMT : Discrete Multitone modulation**

- Channel description:
  - main motivation: the modulators & demodulators in vector coding are channel dependent. If $\overset{ISI}{\text{channel}}$ taps change F & M matrices have to be recomputed

$\left(\frac{DMT}{OFDM}\right)$: Solves this problem

Idea: turn the Toeplitz matrix → circulant matrix by using a cyclic prefix of length ν: $\overset{\text{all}}{\text{circulant}}$ matrices have the same Fourier

$$\underset{\substack{\uparrow\\ N\times 1}}{\begin{bmatrix} y_{N-1} \\ y_{N-2} \\ \vdots \\ y_0 \end{bmatrix}} = \underset{\substack{(N \gg v)\quad\uparrow\\ N\times N \text{ circulant matrix}}}{\begin{bmatrix} p_0 & p_1 & \cdots & p_v & \cancel{p_v} & \cdots \\ 0 & p_0 & \cdots & p_v & \cdots & \cancel{p_v} \\ 0 & & \ddots & & & \\ p_v & & & \ddots & & p_{N-2} \\ & \ddots & & & \ddots & p_{N-1} \\ p_1 & p_2 & \cdots & p_v & \cdots & p_0 \end{bmatrix}} \begin{bmatrix} X_{N-1} \\ X_{N-2} \\ \vdots \\ X_0 \end{bmatrix} \cancel{p_v} + N$$

(each row is obtained by shifting the previous row by one & wrapping around.)

<u>Idea</u>:

$$\underset{\sim}{x} = \begin{bmatrix} X_{N-1} \\ X_{N-2}' \\ \vdots \\ X_0 \\ \boxed{\begin{matrix} X_{-1} \\ \vdots \\ X_{-v} \end{matrix}} \end{bmatrix} \begin{matrix} \\ \\ \\ \\ X_{N-1} \\ X_{N-2} \\ \vdots \\ X_{N-v+1} \end{matrix}$$

Replace by cyclic prefix (instead of setting to 0)

then set [illegible]

$$\boxed{\underset{\sim}{y} = \tilde{P}\underset{\sim}{X} + N.}$$

VERY NEAT IDEA

- Eigenvalue decomposition ~~of~~ of $\tilde{P}$:

[Note that the eigenvalue Decom and SVD of a square matrix are similar except that the ED forces : S1 the matrices to be inverse & hence cannot make $\Lambda \geqslant 0$ always in eigenvalue decomp).

$$\boxed{\tilde{P} = M \Lambda M^*}$$

has this form

where:

$$\boxed{M \text{ is the IDFT matrix.}}$$

$$\Lambda = \begin{bmatrix} \lambda_{N-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_0 \end{bmatrix}$$

λ_i : eigenvalues of $\underline{P}$.

Note : $\underset{\substack{\uparrow \\ \text{eigenvalues}}}{|\lambda_n|} = \underset{\substack{\uparrow \\ \geqslant 0 \text{ (if SVD used)}}}{\underline{\underline{\lambda_{n,SVD}}}}$

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⑤: DFT :

a)

DFT:
$$X_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k \cdot e^{-j \frac{2\pi}{N} kn}.$$

IDFT :
$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k \cdot e^{j \frac{2\pi}{N} kn}$$

b) Let :

$$Q \triangleq \begin{bmatrix} e^{-j\frac{2\pi}{N}(N-1)(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1).1} \\ e^{-j\frac{2\pi}{N}(N-2)(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-2).1} \\ \vdots & & \\ \underbrace{e^{-j\frac{2\pi}{N}.1.(N-1)}}_{\underline{q}_{N-1}} & \cdots & \underbrace{e^{-j\frac{2\pi}{N}.1.1}}_{\underline{q}_0} \end{bmatrix}$$

$$= [\underline{q}_{N-1} \; \cdots \; \underline{q}_0]$$

Then $\underline{X} = \overset{\text{DFT matrix}}{Q}\,\underset{\sim}{x}$

$$\underset{\sim}{x} = \underset{\substack{\uparrow \\ \text{IDFT matrix}}}{Q^*}\,\underline{X}$$

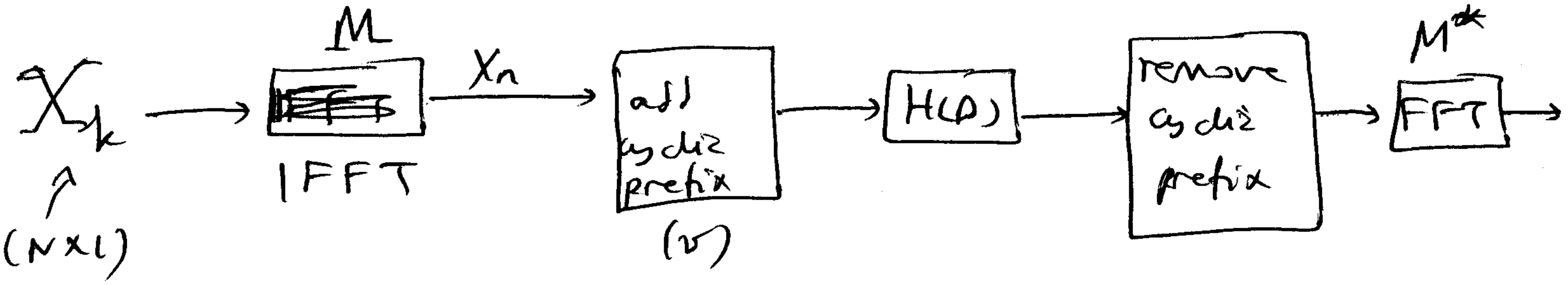
$$QQ^* = Q^*Q = I.$$

c) claim : $\boxed{\tilde{P} q_n = \lambda_n \cdot q_n}$ (Prove that these q's indeed diagonalize the $\tilde{\underline{P}}$ matrix)

Circulant

~~$\tilde{P} q_0 = (\sum p_i)$~~

⑥ DMT system:



$\underline{y} = \Lambda \underline{x}$. (effective system)

⑦ Complexity:

$O(N \log N)$ operations (since $\underline{\underline{FFT}}$ is used)

(VC requires $O(N^2)$)

— also VC requires $(N+\nu)$ vector multiplies.

— pays the price of cyclic prefix.

a) performance of VC (unrestricted) must be $\geq$ DMT

b) penalty on transmit energy.

$$E[|X|^2] = N\bar{\mathcal{E}}.$$

c) overhead $\to 0$ as $N \to \infty$.

$$\lim_{N\to\infty} DMT = VC.$$

⑨ Example:

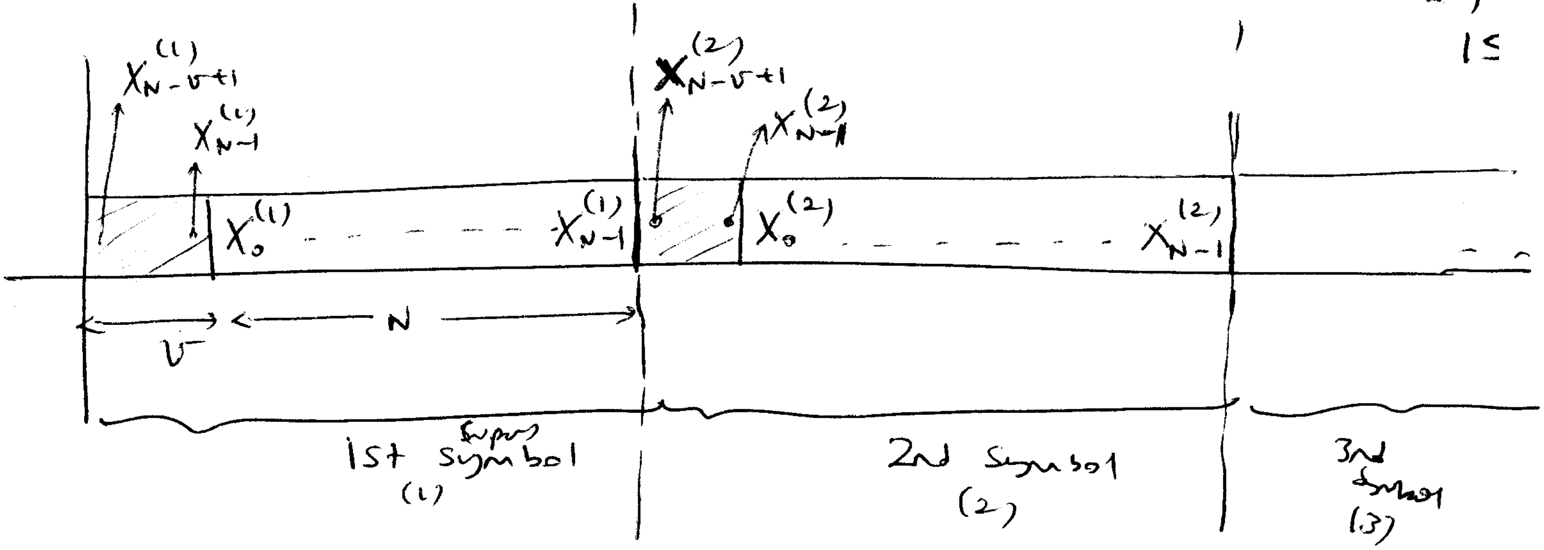
Same $1+0.9D^{-1}$ example

$N = 8$

$\Gamma = 1$ (0dB)

$$K = \frac{1}{7}\left[8 + \sum_{n=0}^{6} \frac{1}{g_n}\right]$$

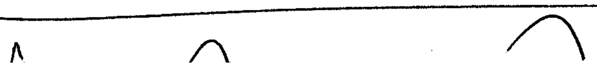
$$SNR_{DMT} = \underline{7.6\ dB} \quad (< \underbrace{8.1dB}_{VC})$$


$X^{(1)}_{N-v+1}$
$X^{(1)}_{N-1}$
$X^{(1)}_{0}$ - - - - $X^{(1)}_{N-1}$
$X^{(2)}_{N-v+1}$
$X^{(2)}_{N-1}$
$X^{(2)}_{0}$ - - - - $X^{(2)}_{N-1}$
v
N
1st super symbol
(1)
2nd symbol
(2)
3rd symbol
(3)

Excess loss $= \dfrac{v}{N+v}$

<u>PAR</u> : peak-to-average ratio problem.

- Since ~~modulated~~ original symbols in freq domain X_k.
  can have large peaking signals in time domain
  → need to have expensive power amplifiers
  that ~~keep the signal~~ are linear
  over a larger range
- Some Solutions:
  - make some of the carriers "dummy"
    and change their energies to minimize PAR
    at output of amplifiers



- Question very long channels.

e.g. $H(D) = \frac{1}{1+0.9D}$

∴ — use equalizer + DMT.

(Brings the channel ~~to~~ from $\nu \longrightarrow \nu'$ taps.

Then DMT can be used ~~to~~ with a cyclic prefix of ν'.)