LECTURE #4 A

- We shall continue with the analysis of trunking systems.

- Last time, we evaluated the blocking probability of a LCC (lost call dropped) system, and obtained the Erlang B formula as the result.
- This time, we will analyze the performance of the LCD (lost call delayed) system.

  In the last lecture, we formulated this as a Markov process and found the stationary probability distribution π_i.

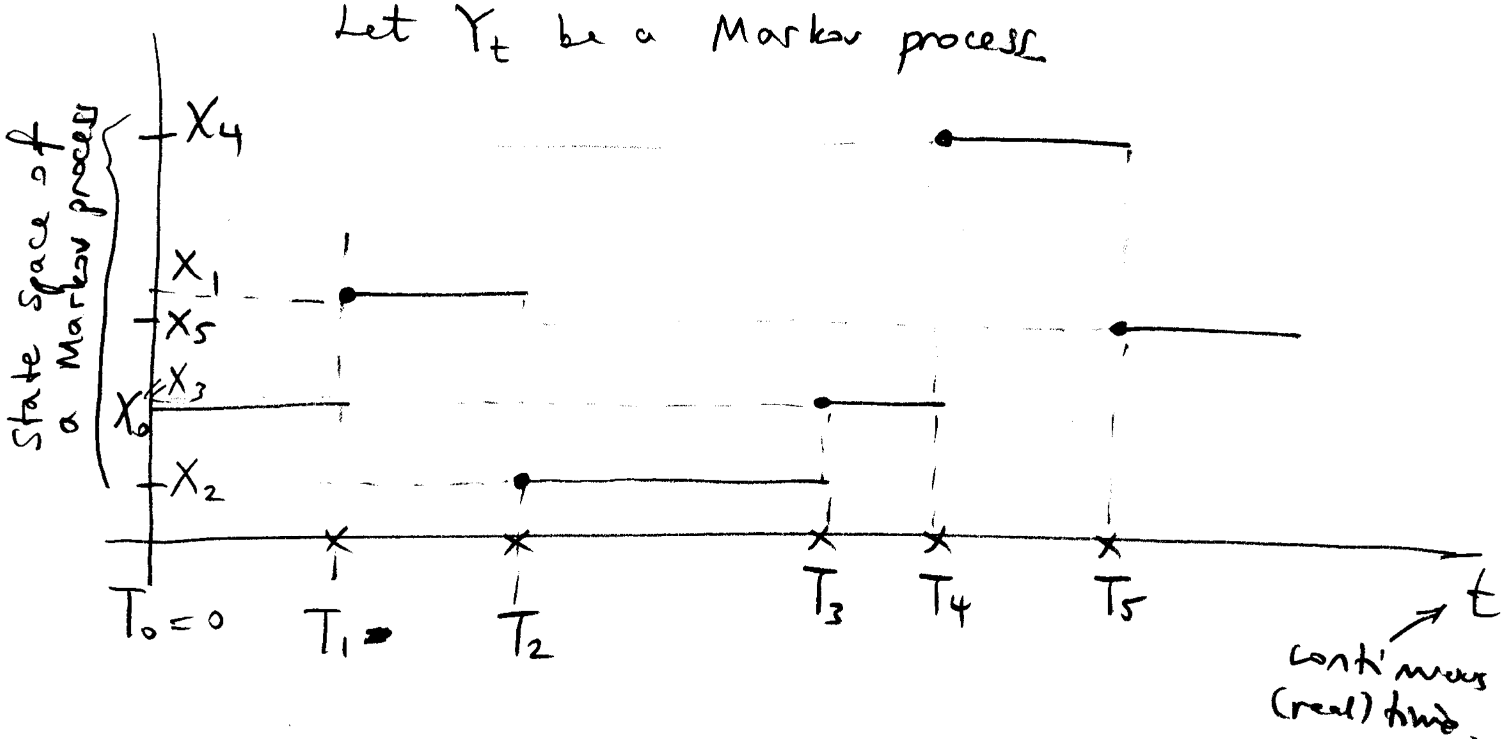
  As performance measures, we will evaluate:

1) $P\{\text{delay} > 0\}$, i.e. probability that a placed call is delayed, and
2) the <u>average delay</u> of a call (any call placed.)

To understand / derive these, we need to review some results from the theory of Markov processes.

Structure of a Markov process:

Let Y_t be a Markov process



- Let $\{T_i\}$ denote the times when the process transitions to a new state. $(T_0 = 0)$.
- Let W_t denote the waiting time in a state until a change of state (transition) occurs, i.e.,

$$T_0 = 0, \qquad T_{n+1} - T_n = W_{T_n}, \quad n \in \mathbb{N} \text{ (~~positive~~ non-negative integers)}$$

- Let X_n denote the state at time T_n.

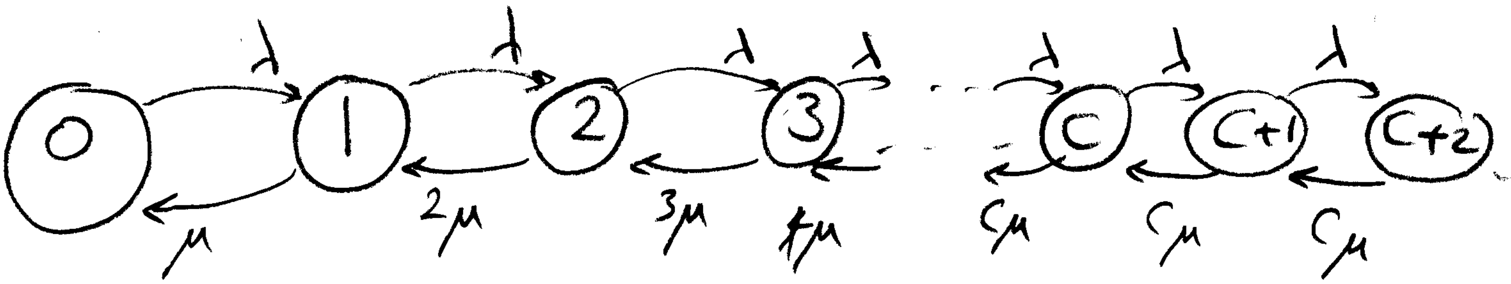
$$X_n = Y(X_n)$$

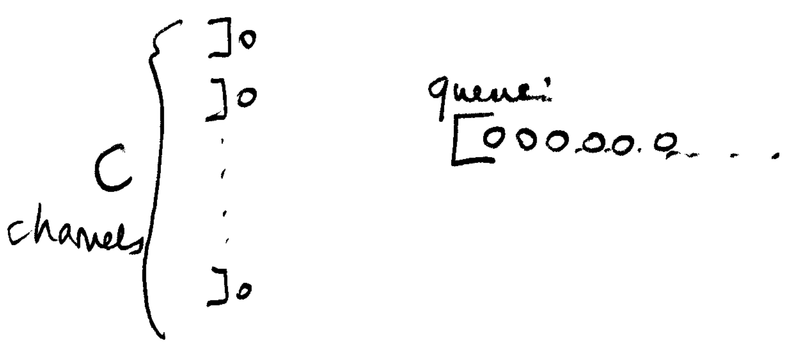
* <u>THEOREM</u>: $\{X_n ; n \in \mathbb{N}\}$ is a Markov chain, and

$T_{n+1} - T_n$ has an exponential distribution with a parameter that depends on X_n.

[— That is: The set of states, viewed in discrete-time form a Markov chain, and the sojourn times (waiting times) of the Markov process are exponentially distributed that depends only on the <u>state</u> in which the sojourn occurs.]

- Let's go back to our model: (LCD):





(a) $P\{$a call is delayed$\}$ = $P\{$system is in one of the states $c, c+1, c+2 \ldots\}$

$$= \sum_{i=c}^{\infty} \pi_i \,,$$

$$= \sum_{i=c}^{\infty} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{c^{i-c}}\,\pi_0$$

See p.609 Rappaport

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \cdot \frac{1}{\left(1-\frac{\lambda}{\mu c}\right)}\,\pi_0$$

(valid for $\frac{\lambda}{\mu c} < 1$)

Substitute for π_0

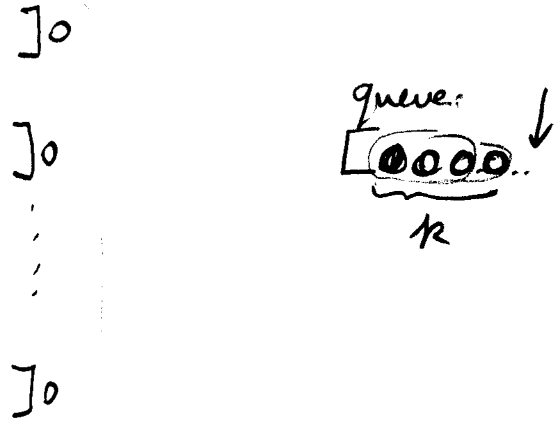
$$= \frac{\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c}{\left(1-\frac{\lambda}{\mu c}\right)\left[\sum_{i=0}^{c-1}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{1}{\left(1-\frac{\lambda}{\mu c}\right)}\right]}$$

Recall: $A \equiv \frac{\lambda}{\mu}$ $(= \lambda H)$ $(H \equiv 1/\mu)$

$$= \boxed{\frac{A^c}{A^c + c!\left(1-\frac{A}{c}\right)\sum_{i=0}^{c-1}\frac{A^i}{i!}}}$$

"Erlang C formula"

(b) If a call is delayed, how long will it wait in the queue?



Let W denote the time that a call has to wait given that it is delayed. Then

$$P\{W > t \mid \text{call is delayed}\} = \sum_{k=0}^{\infty} P\{W > t, \exists\, k \text{ calls before this one in queue} \mid \text{call is delayed}$$

Assume that there are $k \geq 0$ uses in queue before this call.

The transitions times in the Markov process are times of regeneration. Hence

$$W\Big|_{\substack{\text{these}\\\text{conditions}}} = \underbrace{\text{Service time until at least a user occupying a chanel finishes}}_{\exp(c\mu)} + \underbrace{\text{Service time with } k-1 \text{ calls in queue}}_{\exp(c\mu)} + \cdots + \text{Service time with } 0 \text{ in queue}$$

$= (k+1)\tilde{W}$ where $\tilde{W} \sim \exp(c\mu)$

Then $\overline{W}_{|.} \sim \text{Gamma}(c\mu, k+1)$.

$$P\{W > t \mid \text{call is delayed}\} = \sum_{k=0}^{\infty} P\{W > t \mid \exists\, k \text{ calls in queue}\}\, P\{k \text{ calls in queue}\}$$

given that

$$= \sum_{k=0}^{\infty} \left[\int_t^{\infty} \frac{(c\mu) e^{-c\mu t'} (c\mu t')^k}{k!}\, dt'\right] \cdot \pi_{c+k}$$

$$\pi_{c+k} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{k+c} \frac{1}{c^k}\, \pi_0$$

$$= \frac{1}{c!}\pi_0 \cdot c\mu \int_t^{\infty} dt' e^{-c\mu t'} \underbrace{\sum_{k=0}^{\infty} \frac{(c\mu t)^k}{k!} \cdot \left(\frac{\lambda}{\mu}\right)^{k+c} \cdot \frac{1}{\cancel{c^k}}}_{\sum_{k=0}^{\infty} \frac{\cancel{\mu^k} t^k \lambda^k}{k!\,\cancel{\mu^k}} \frac{\lambda^c}{\mu^c}}$$

$$\cancel{c^k}(\mu t)^k \qquad \frac{\lambda^k \lambda^c}{\mu^{k+c}}$$

$$= \frac{1}{c!}\pi_0\, c\mu \left(\frac{\lambda}{\mu}\right)^c \int_t^{\infty} dt' e^{-c\mu t'} \cdot \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}$$

Recall:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$= \frac{\pi_0 \cdot \lambda^c}{(c-1)!\, \mu^{c-1}} \int_t^\infty dt'\, \underbrace{e^{-c\mu t'}\, e^{+\lambda t'}}_{}$$

$$\underbrace{\frac{1}{c\mu-\lambda}\, e^{-(c\mu-\lambda)t'}}_{e^{-(c\mu-\lambda)t} \cdot (c\mu-\lambda)} \cdot (c\mu-\lambda)$$

$$= \boxed{\frac{\pi_0 \cdot \lambda^c \cdot (c\mu-\lambda)}{(c-1)!\, \mu^{c-1}}}\, e^{-(c\mu-\lambda)t}.$$

must equal $P\{W>0\}$ ($= P\{$call is delayed$\}$)

Hence:

$$\boxed{P\{W>t \mid \text{call is delayed}\} = P\{\underbrace{\text{call is delayed}}_{(W>0)}\} \cdot e^{-(c\mu-\lambda)t} \qquad t>0.}$$

i.e. the waiting time of a call, given that it is delayed, is <u>exponentially distributed</u> with parameter $(c\mu-\lambda)$.

$$\sim \exp(c\mu-\lambda).$$

avoids queues that over time $\to +\infty$ ← (assumption: $c\mu>\lambda$, $c>\lambda/\mu$)

System Design:

- Erlang B: For blocking prob: 2% (0.02)

C	$\max(\lambda/\mu)$
2	$\simeq 0.2$
3	$\simeq 0.6$
10	5
20	$\simeq 10-15$
35	20.

triples

Issues to think about:

1 – which system is better (LCC or LCD) and under what circumstances?

2 – For a fixed probability (of blocking or being delayed), and a fixed C, which one can tolerate a higher traffic intensity?

3 – give practical guidelines about which one to use under what circumstances.

(Hint: Examine the plots p. 81–82 to test your intuitions)

(

Take: traffic intensity (λ/μ) = 10 Erlangs

For C = 15 channels:

LCC: $P_{blocking} = 0.02$

LCD: $P_{delayed} = 0.1$ ($\gg 0.02$)

much higher than 0.02.

Why? — Because when you allow calls to be queued, you shift some positive probability to the states $(C+1), \ldots \infty$; hence when a user arrives, more likely that it will find the system in $C+1, \ldots \infty$ (than ϕ in LCC system)

- Then, the next question is: Is the delay tolerable?
  Because if the delay is too large, then a user would actually hang up (not modeled in our model) and try the call again.
  - So somehow the delay must be tolerable. for LCD to be used.

- compute: $E[W] \doteq \underbrace{P\{\text{delay} > 0\}}_{0.1} \cdot \dfrac{1}{C\lambda - \mu}$.

Average delay:

$$E[W] = \underbrace{\Pr\{\text{delay} > 0\}}_{0.1} \cdot \frac{1}{c\mu - \lambda}$$

$E[\exp_{r.v.}(\sim (c\mu - \lambda))]$

$$\frac{1}{c\mu-\lambda} = \frac{1/\mu}{c - \lambda/\mu} = \frac{H}{C-A}$$

$C = 15$, $A = 10$, $H = 5$ minutes (say).

(average holding time of each call)

$$= 0.1 \cdot \frac{5 \text{ minutes}}{5} = 0.1 \text{ minutes}$$

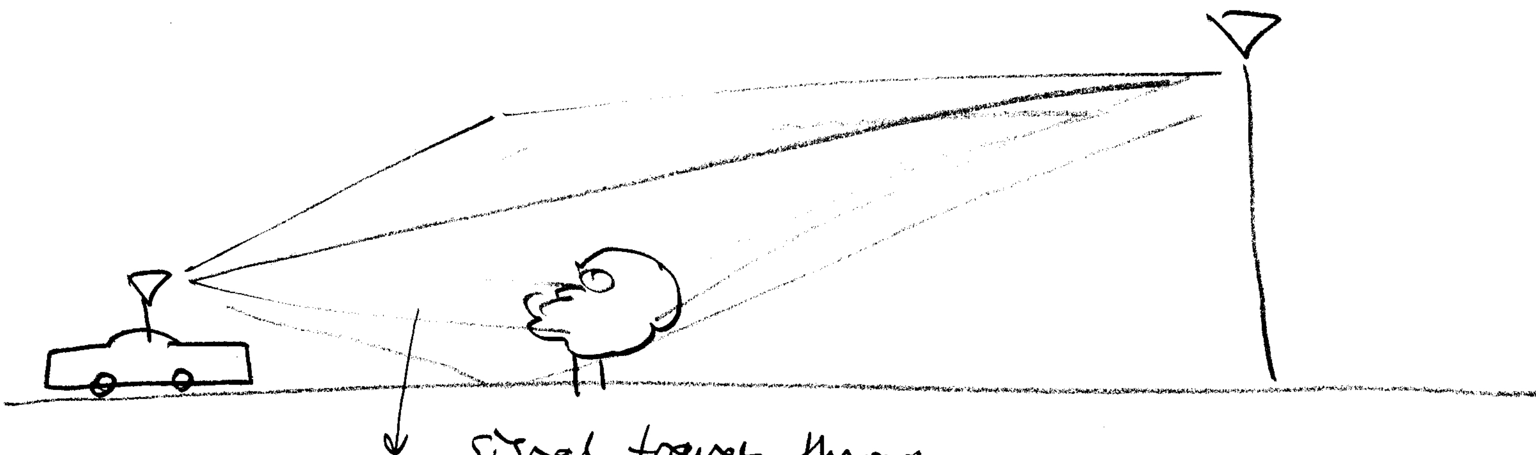
$$= \underline{\underline{6 \text{ seconds}}}$$

(Seems a bit large.. $\simeq$ ~~4~~ 3 seconds seems reasonable in current system)

LECTURE #4B

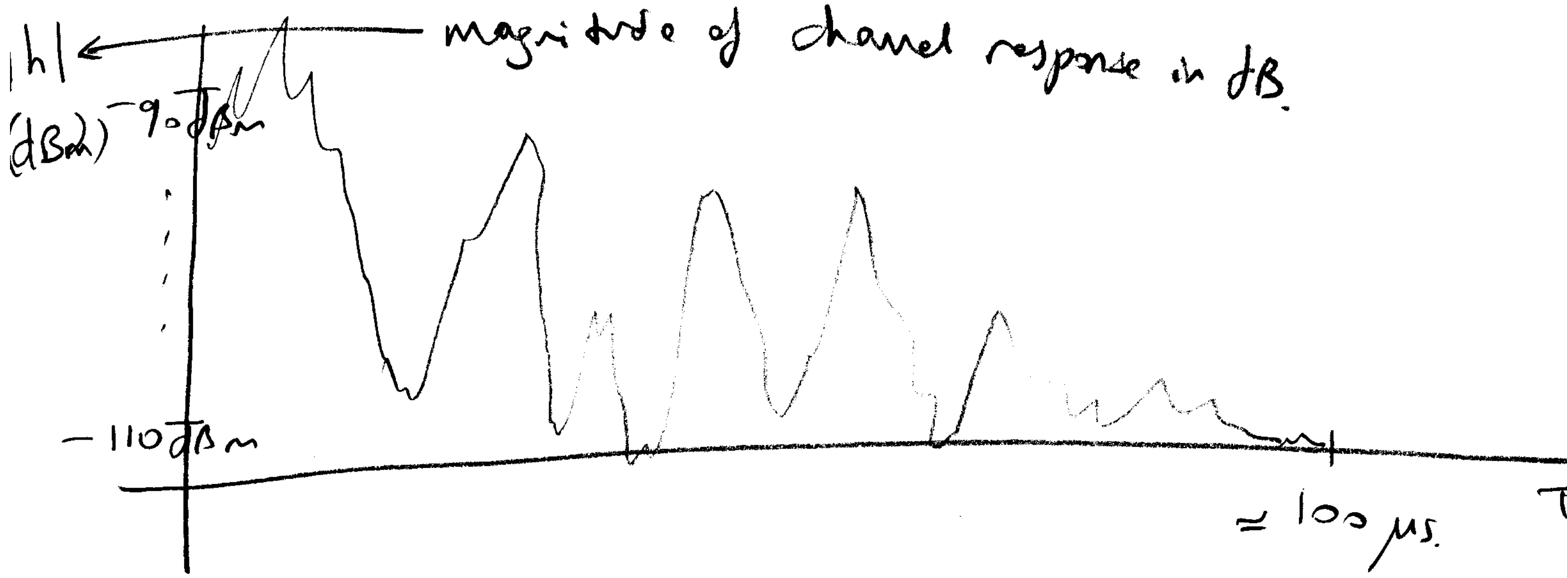
SMALL-SCALE VARIATIONS:

- multipath profile } different effects
- Doppler shifts

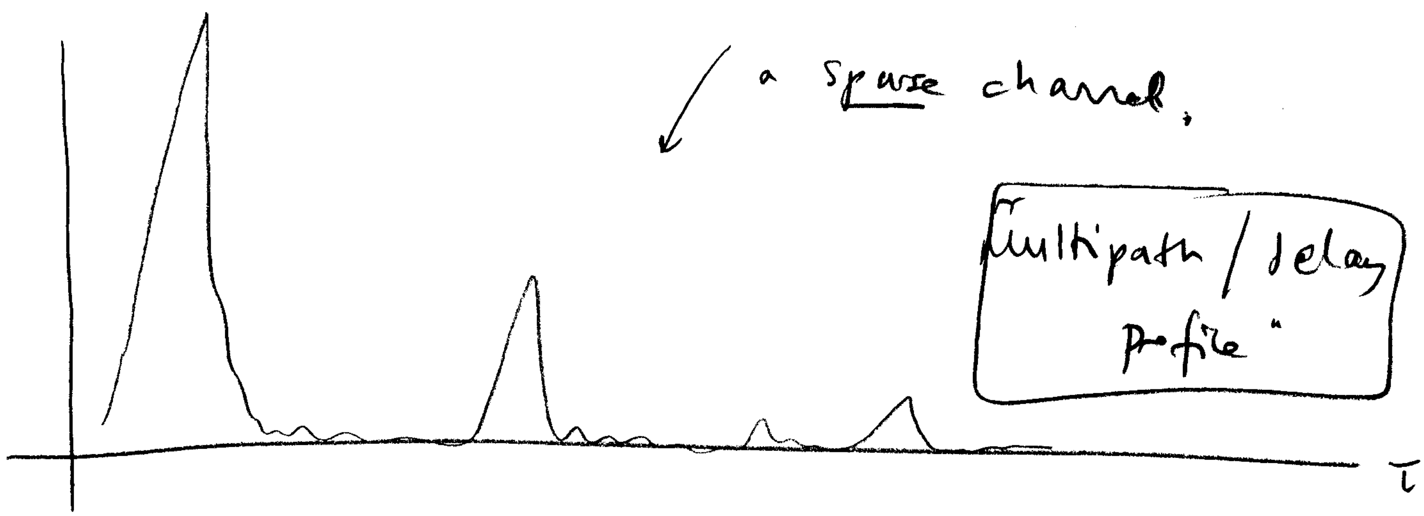


- Assume for now that the user (car) is stationary. (and of course BS stationary).
  - we will approximate the channel in between as stationary. (But over the long run, the large-scale variations of obstacles causes the channel to change), So: we take an interval like several seconds over which the channel is constant in time.

- But of course there is still <u>multipath</u>!



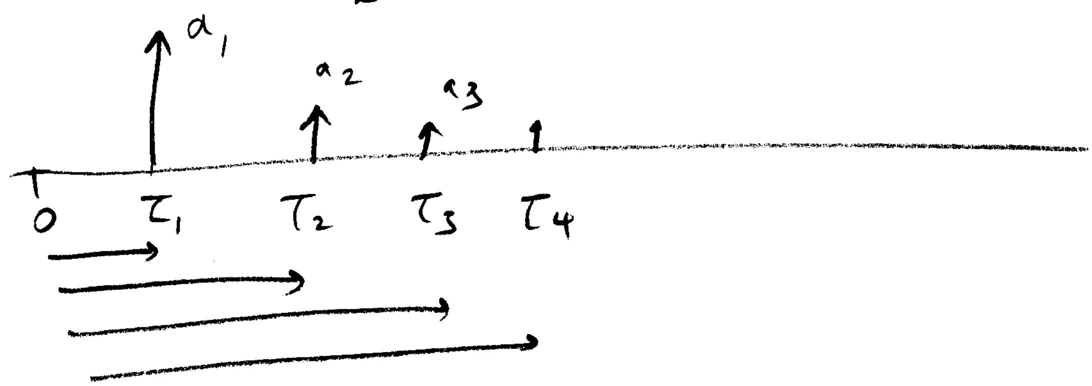
- Send an impulse to the channel and measure its response



- <u>delay spread</u>: is an average measure of how far the energy of the signal ~~expands~~ extends <u>around the mean</u>.

- **Mean excess delay:**

$$\bar{\tau} = \frac{\sum_k a_k^2 \tau_k}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\cdot \tau_k}{\sum_k P(\tau_k)}$$



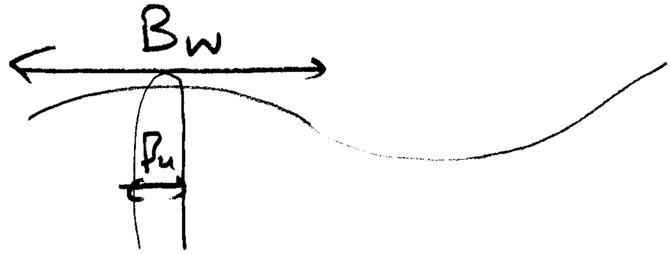
- **Rms delay spread:**

$$\sigma_\tau = \sqrt{\overline{\tau^2} - (\bar{\tau})^2}$$

where: $$\overline{\tau^2} = \frac{\sum_k a_k^2 \tau_k^2}{\sum_k a_k^2} = \frac{\sum P(\tau_k)\,\tau_k^2}{\sum_k P(\tau_k)}$$

Env.	Freq.	σ_τ
urban	$\simeq$ 900 MHz	1.3 – 3.5 μs (NY) 10 – 25 μs (SF)
Suburban	$\simeq$ 900 MHz	200 – 310 ns (why smaller? than urban.) 1960 – 211 ns (1.96 – 2.11 μs)
indoor	1500 MHz	10 – 50 ns
indoor	850 MHz	270 ns
indoor	1.9 GHz	70 – 94 ns

Coherence bandwidth: amount of BW over which the freq. response of channel does not vary (much).



If $B_u \ll B_w$: flat channel

If $B_u \gg B_w$: freq.-selective channel

B_u — input bandwidth

Roughly:
$$\boxed{B_c \approx \frac{1}{\sigma_\tau}}$$

σ_τ — delay spread.

- equalization techniques — OFDM