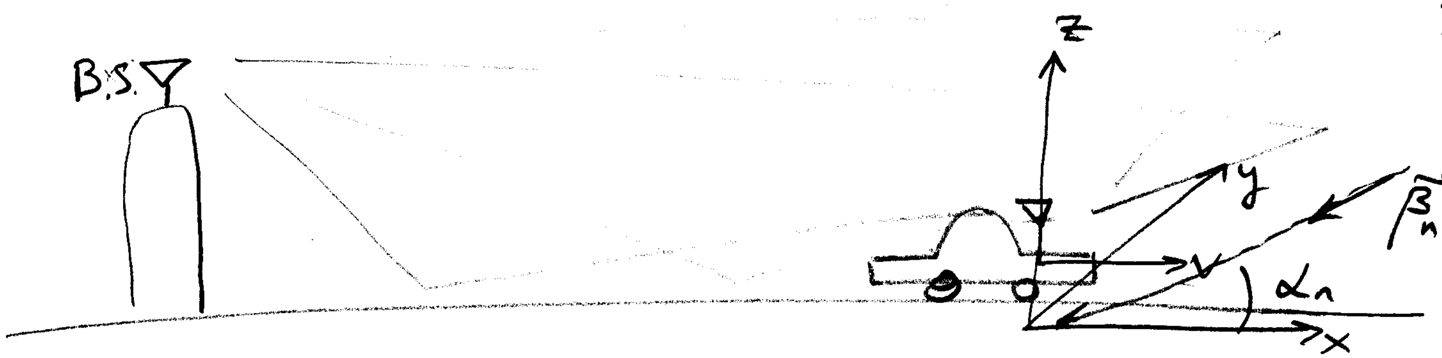
LECTURE 6

- We will continue by examining the spectral broadening (Doppler spread) that occurs due to the fact that different paths arriving at different angles (incident on the mobile user antenna) experience <u>different</u> Doppler shifts.

- A common analytical model for the spectrum of the received signal was derived by Gans and is based on Clarke's model.

<u>AIM</u>: To derive the power spectral density (of the electric field of) the received signal. [Note that this is a function of our frequency ν — but it is common to use f as the frequency variable — you should keep it straight in your mind which frequency domain we are in, even though f may be used for both.]



<u>Recall Clarke's model</u>:

1 No LOS component.

2 B.S. is stationary.

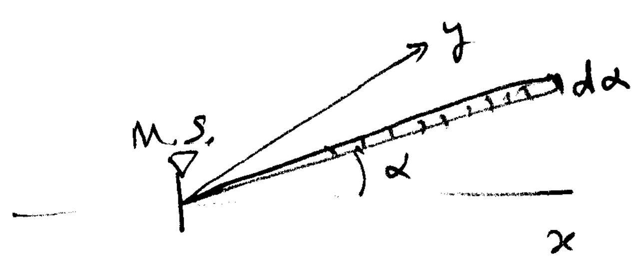
3 The transmitted wave is <u>vertically polarized</u>. (i.e. its $\vec{E}$ field is pointing in the z direction — it has only an E_z component.)

[Since the direction of propagation ($\vec{\beta}$) must be $\perp$ to $\vec{E}$, the $\vec{\beta}$ vector must lie on the xy plane.]

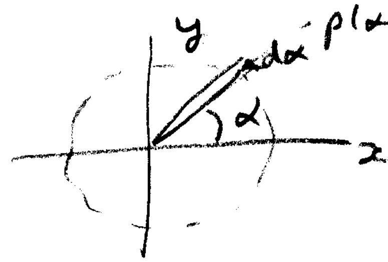
4 We assume that the nth (multipath) component arrives at (i.e. incident on) the mobile antenna at an angle of α_n.

5 Assume that N (# of multipath components) is large.

<u>Gans's derivation of the power spectral density of $\vec{E}$</u>



Power arrives at (is incident on) the M.S antenna from directions $\alpha \in [0, 2\pi)$.



The fraction of the power received at angle α, will be denoted by $p(\alpha)$.

- Then, the amount of power incident within an differential of $d\alpha$ around α is $p(\alpha) . d\alpha$.

• Let A be defined as the total power that would be received by an isotropic antenna, i.e.

$$A \equiv \frac{P_r}{\int_0^{2\pi} p(\alpha)\, d\alpha}$$

P_r ← total received power from all directions.

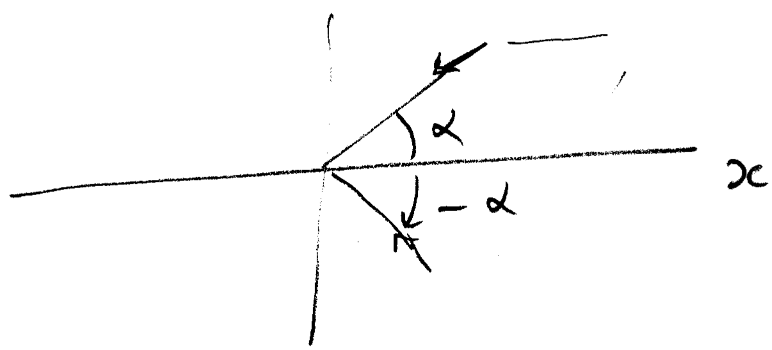
(i.e. A will serve as a normalizing constant)

- Then, for an M.S. antenna with gain pattern $G(\alpha)$,

(e.g. )

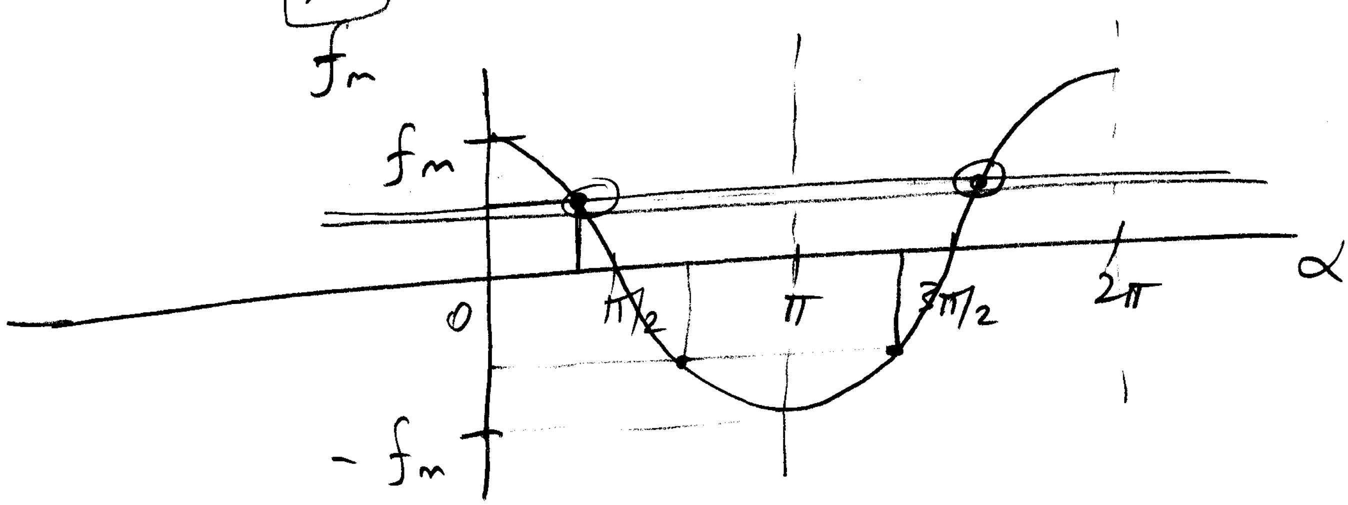
$$P_r = \int_0^{2\pi} A\, G(\alpha)\, p(\alpha)\, d\alpha$$

KEY (in the analysis): There is a <u>one-to-one</u>



relationship between the Doppler shift ($f(\alpha)$) of a wave and α, in the range $\alpha \in [0, \pi)$.

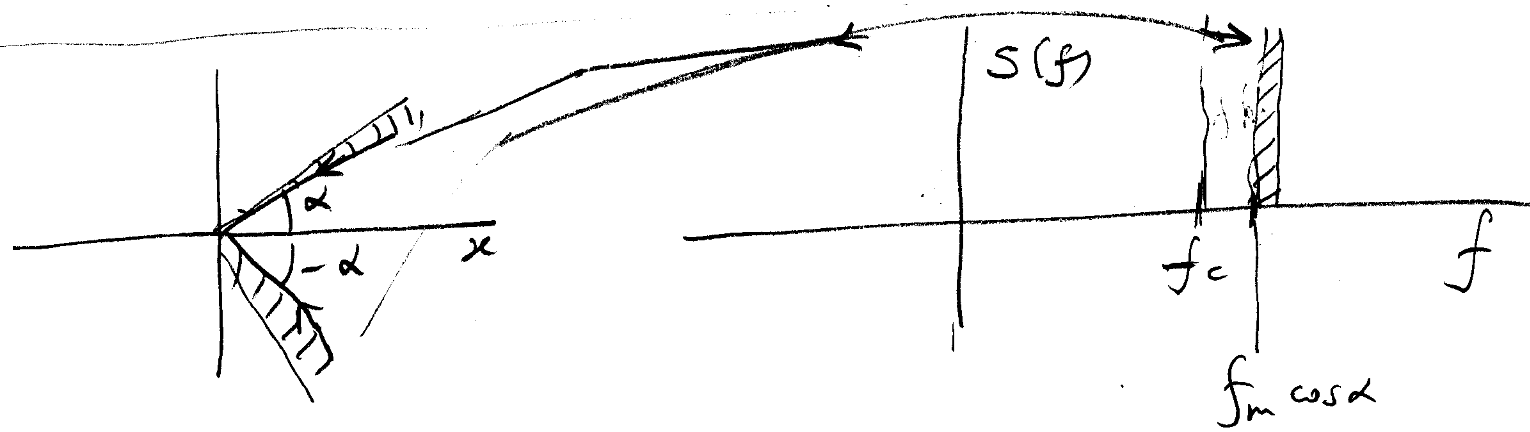
$$f(\alpha) = \underbrace{\boxed{\frac{v}{\lambda}}}_{f_m} \cos(\alpha) + f_c$$



In the range $\alpha \in [0, 2\pi)$, we see that a Doppler shift of $f(\alpha)$ is experienced only by waves incident at an angle of $\underline{\underline{\alpha}}$ or $\underline{\underline{-\alpha}}$. [$f(\alpha)$ is an even function]

Then, we can set up a relationship between the amount of power that falls in the range $[f, f+df]$ and the amount of power that comes in from the angular range $[\alpha, \alpha+d\alpha]$.

$$\boxed{S(f)|df| = A\left[p(\alpha)\,G(\alpha) + p(-\alpha)\,G(-\alpha)\right]|d\alpha|} \quad (0)$$



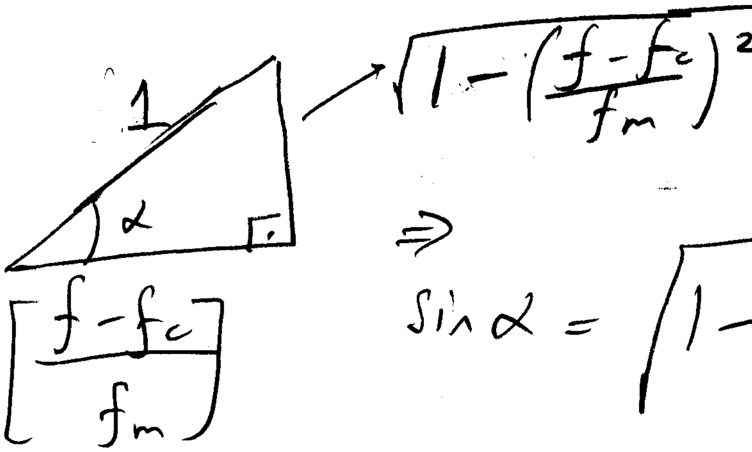
Now, $f(\alpha) = f_m \cos\alpha + f_c$

$$\frac{df(\alpha)}{d\alpha} = -f_m \cdot \sin\alpha.$$

$$\Rightarrow \boxed{|df(\alpha)| = f_m \cdot |\sin\alpha| \cdot |d\alpha|} \quad (1)$$

In addition $f(\alpha) = f_m \cos\alpha + f_c$

$$\Rightarrow \frac{f(\alpha) - f_c}{f_m} = \cos\alpha \Rightarrow \boxed{\alpha = \cos^{-1}\left[\frac{f(\alpha) - f_c}{f_m}\right]} \quad (2)$$



$\Rightarrow$

$$\sin\alpha = \sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}.$$

$\left[\text{If } \left(\frac{f-f_c}{f_m}\right)^2 \le 1\right]$

Then, from (1) above

$$|df(\alpha)| = f_m \cdot \left|\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}\right| \cdot |d\alpha|$$

Substituting into (0) gives:

$$\boxed{S(f) = \frac{A\left[p(\alpha)\,G(\alpha) + p(-\alpha)\,G(-\alpha)\right]}{f_m\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}}, \quad \text{for: } \left(\frac{f-f_c}{f_m}\right)^2 \le 1 \quad [\Leftrightarrow |f-f_c| \le f_m]}$$

This is a general expression that relates the $S(f)$ (power spectral density) to the $p(\alpha)$, $G(\alpha)$, and f_c, f_m.

$[S(f) = 0$ outside the range $|f-f_c| \le f_m]$

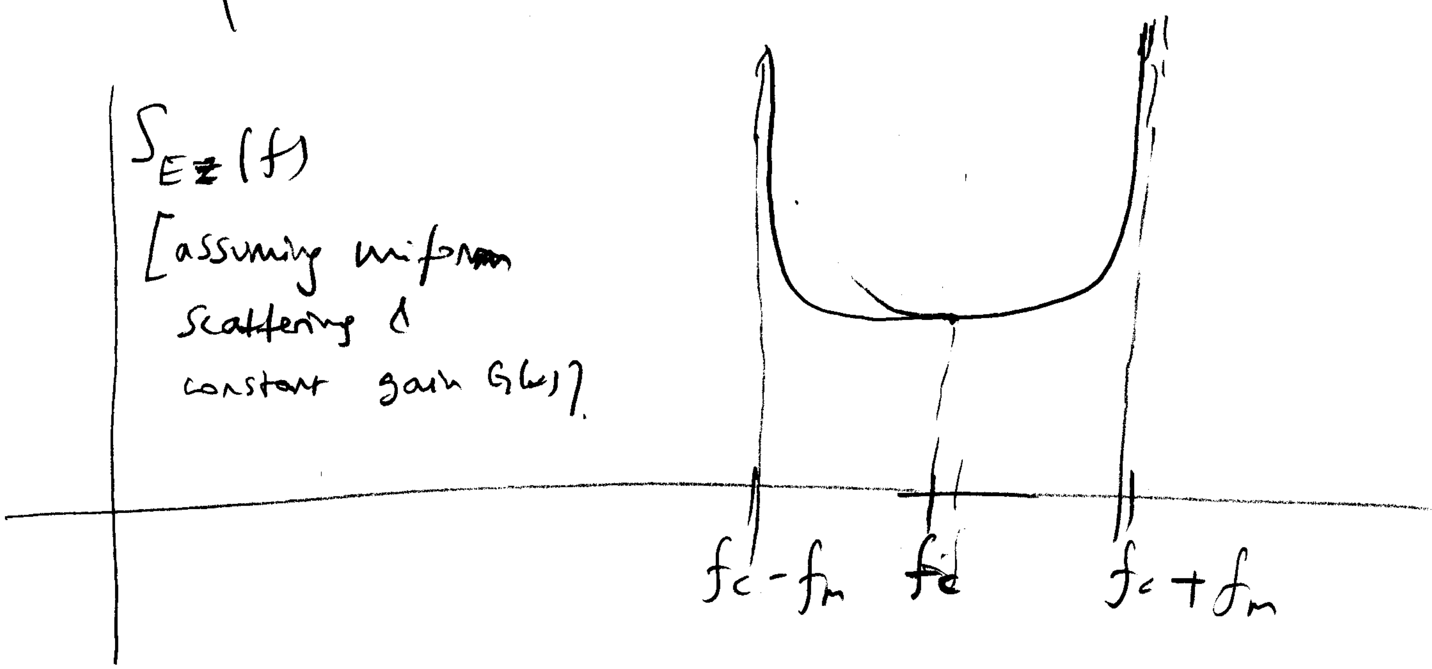
- Now, if we assume a <u>uniform</u> distribution on $p(\alpha)$
[i.e. power arrives equally from all directions α.],

then: $p(\alpha) = \frac{1}{2\pi}$ $\alpha \in [0, 2\pi)$, and

- if we assume $G(\alpha) = \text{constant} = c$, then

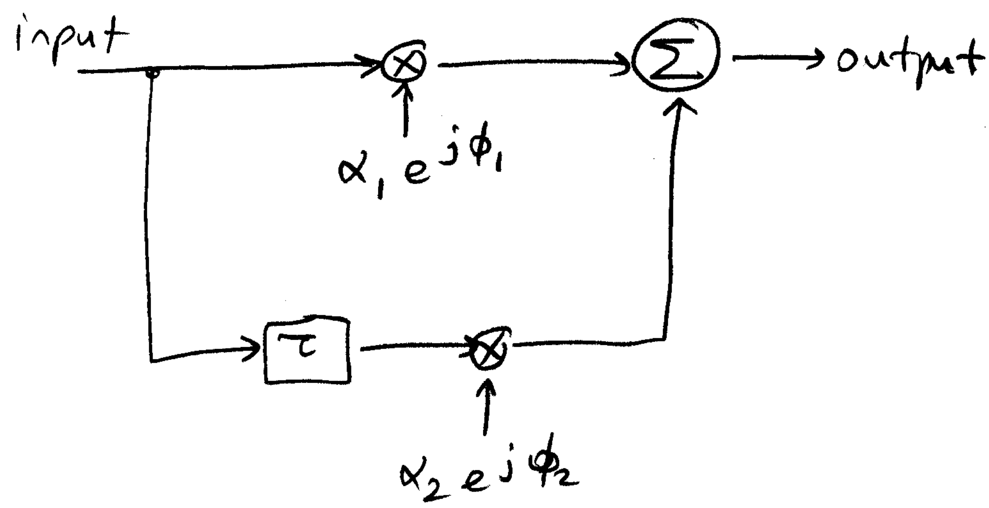
$$S(f) = \begin{cases} \dfrac{\frac{Ac}{2\pi}}{f_m \sqrt{1 - \left(\frac{f - f_c}{f_m}\right)^2}}, & |f - f_c| < f_m \\ 0 & \text{else} \end{cases}$$

(S_{E_z})



Two-ray Rayleigh fading model

[the simplest model that combines multipath delay profile and the fading due to mobility.]



$$\underbrace{h_b(t)}_{\text{baseband impulse response}} = \alpha_1 e^{j\phi_1} \delta(t) + \alpha_2 e^{j\phi_2} \delta(t-\tau)$$

where $\alpha_1 \perp\!\!\!\perp \alpha_2$, and $\alpha_i \sim \text{Rayleigh}(\sigma_i)$, $i \in \{1,2\}$ and

$\phi_1 \perp\!\!\!\perp \phi_2$, and $\phi_i \sim \text{Uniform}[0, 2\pi)$.

τ : deterministic.

[$\alpha_2 = 0$ gives a Rayleigh fading channel.]

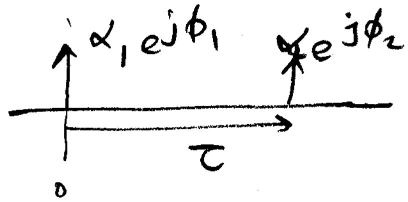
- By varying τ, can generate different 2-path freq. selective models.

[Exercise: simulate this channel; later, we will describe the performance of modulation schemes on this channel model.]

Questions:

1 – What is the coherence bandwidth of the above channel?

Rough Answer:



[See our statistical channel model definitions starting from p. 12 & get exact answer.]

$$\sigma_\tau = \sqrt{\overline{\tau^2} - \overline{\tau}^2}$$

$$\overline{\tau} = \frac{\sum_k \tau_k P(\tau_k)}{\sum_k P(\tau_k)} = \frac{\tau_1 \cdot P(\tau_1)}{P(\tau_0) + P(\tau_1)} \qquad (\text{where } \tau_1 = \tau)$$

$$P(\tau_1) = E\left[|\alpha_1 e^{j\phi_1}|^2\right]$$

$$= E\left[|\alpha_1|^2\right]$$

$$= E[\alpha_1^2] \qquad (\text{since } \alpha_i \sim \text{Rayleigh} \Rightarrow \alpha_1 \geq 0)$$

$$= 2\sigma_1^2 \qquad \text{Recall: } \left[E(\alpha_i^2) = 2\sigma_i^2,\ E(\alpha_i) = \sigma_i\sqrt{\pi/2}.\right]$$

$$P(\tau_0) = 2\sigma_0^2$$

Hence, $$\overline{\tau} = \frac{\tau \cdot 2\sigma_1^2}{2(\sigma_0^2 + \sigma_1^2)} = \frac{\tau \cdot \sigma_1^2}{\sigma_0^2 + \sigma_1^2} = \frac{\tau}{(\sigma_0/\sigma_1)^2 + 1}$$

[If $\sigma_1 = \sigma_2$, then $\overline{\tau} = \frac{2\tau \cdot \cancel{\sigma^2}}{4\cancel{\sigma^2}} = \frac{\tau}{2}$.]

$$\overline{\tau^2} = \frac{\sum_k \tau_k^2 \cdot P(\tau_k)}{\sum_k P(\tau_k)} = \frac{\tau_1^2 \cdot P(\tau_1)}{P(\tau_0) + P(\tau_1)}$$

$$= \frac{\tau^2 \cdot 2\sigma_1^2}{2(\sigma_0^2 + \sigma_1^2)} = \frac{\tau^2 \cdot \sigma_1^2}{\sigma_0^2 + \sigma_1^2}$$

$$= \frac{\tau^2}{1+\left(\frac{\sigma_0}{\sigma_1}\right)^2}$$ 1.

[If $\sigma_1 = \sigma_2$, $\overline{\tau^2} = \frac{\tau^2}{2}$.]

Then:

$$\sigma_\tau = \sqrt{\frac{\tau^2}{\left[1+\left(\frac{\sigma_0}{\sigma_1}\right)^2\right]} - \frac{\tau^2}{\left[\left(\frac{\sigma_0}{\sigma_1}\right)^2+1\right]^2}}$$ Let $\beta \equiv \left(\frac{\sigma_0}{\sigma_1}\right)^2$

$$1\cancel{+}\beta - \cancel{1} = \beta$$

$$= \tau \cdot \sqrt{\frac{\beta}{(1+\beta)^2}} = \tau \sqrt{\frac{(\sigma_0/\sigma_1)^2}{\left(1+(\sigma_0/\sigma_1)^2\right)^2}}$$

$$= \tau \cdot \left[\frac{(\sigma_0/\sigma_1)}{1+(\sigma_0/\sigma_1)^2}\right]$$

[If $\beta = 1$, $\sigma_\tau = \tau/2$]

Then

$$\boxed{B_c \cong \frac{1}{\sigma_\tau} = \frac{1}{\tau} \cdot \frac{1+(\sigma_0/\sigma_1)^2}{(\sigma_0/\sigma_1)} \text{ Hz.}}$$

[$B_c = \frac{2}{\tau}$ Hz when $\sigma_0 = \sigma_1$].

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2 _ What is the <u>coherence time</u> of this channel?

- So far, we talked about the coherence time of a <u>deterministic</u> channel model. With statistical fading models, we need a generalization of the coherence time concept.

Statistical channel models:

Let $h(t;\tau)$ be a random process (indexed by t and τ).

In general, $h(t;\tau)$ may be a complex random process (e.g. the baseband equivalent channel of a passband signal.)

- If we assume that $h(t;\tau)$ is wide-sense stationary (in t), then, it has an autocorrelation function

$$\boxed{\phi_c(\tau_1,\tau_2;t) \equiv \frac{1}{2} E[h^*(\tau_1;t)\, h(\tau_2;t+\Delta t)]}$$

definition (for a complex r.p.)

- Further, if the scattering environment at the two different delays is uncorrelated (i.e. if the channel response at τ_1 is uncorrelated with that at τ_2 as in the 2-ray model), then:

$\perp[\phi_1 \perp\!\!\perp \phi_2$ and $\alpha_1 \perp\!\!\perp \alpha_2]$

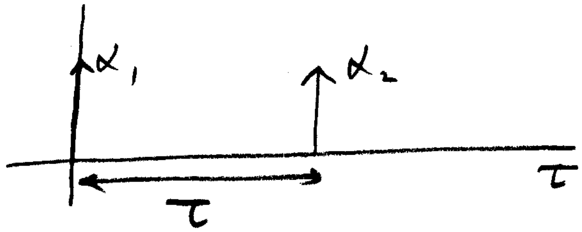
$$= \phi_c(\tau_1;\Delta t)\, \delta(\tau_1-\tau_2)$$

$$\equiv \boxed{\phi_c(\tau;\Delta t)}$$

average power output as a function of the delay τ, at time instant Δt.

For the 2-ray model above:

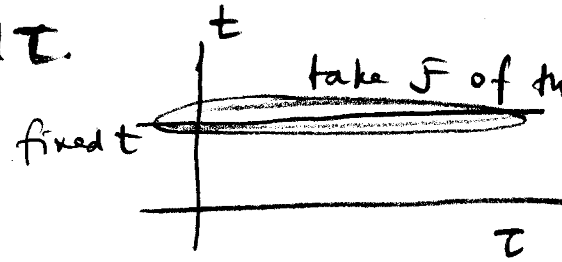
$\phi_c(\tau, 0)$:



[Further, $\phi_c(\tau; \Delta t)$ may be equal to $\phi_c(\tau, 0)$ if the channel is constant (i.e. T_c = coherence time $= +\infty$)]

Now, the

$$H(f;t) \equiv \int_{-\infty}^{+\infty} h(\tau;t)\, e^{-j2\pi f \tau}\, d\tau.$$



Remark.

[If $h(t;\tau)$ is a complex Gsn r.p. in t variable,

Then $H(f;t)$ is also a complex Gsn r.p. in t variable.]

Now,

$$\boxed{\Phi_c(f_1, f_2; \Delta t) \equiv \frac{1}{2} E[H^*(f_1;t)\, H(f_2; t+\Delta t)]}$$

Relationship between $\phi_c(\tau; \Delta t)$ and $\Phi_c(f_1; f_2; \Delta t)$

$$\Phi_c(f_1, f_2; \Delta t) = \frac{1}{2} E\left\{\left[\int_{-\infty}^{+\infty} h^*(\tau;t)\, e^{+j2\pi f_1 \tau}\, d\tau\right] \cdot \left[\int_{-\infty}^{+\infty} h(\tau'; t+\Delta t)\, e^{-j2\pi f_2 \tau'}\, d\tau'\right]\right\}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\left[h^*(\tau_1; t)\, h(\tau_2; t+\Delta t)\right] e^{j2\pi(f_1\tau - f_2\tau')}\, d\tau\, d\tau'$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi_c(\tau_1;\Delta t)\, \delta(\tau_1-\tau_2)\, e^{j2\pi(f_1\tau - f_2\tau')}\, d\tau\, d\tau'$$

$$= \int_{-\infty}^{+\infty} \phi_c(\tau_1;\Delta t)\, e^{j2\pi(f_1-f_2)\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \phi_c(\tau_1;\Delta t)\, e^{-j2\pi(\Delta f)\tau}\, d\tau$$

where $\Delta f = f_2 - f_1$.

Hence:

$$\boxed{\underbrace{\phi_c(\tau;\Delta t)}_{\text{multipath intensity profile (of a statistical channel)}} \xleftrightarrow{\mathcal{F}} \underbrace{\Phi_c(\Delta f;\Delta t)}_{\text{spaced-freq., spaced-time correlation fn of the channel}}}$$

(due to uncorrelated scattering) assumption

$$\sigma_\tau = \sqrt{\frac{\int_0^\infty (\tau-\bar{\tau})^2 \phi_c(\tau)\, d\tau}{\int_0^\infty \phi_c(\tau)\, d\tau}}$$

rms delay spread

for a ~~[illegible]~~ random WSS channel model

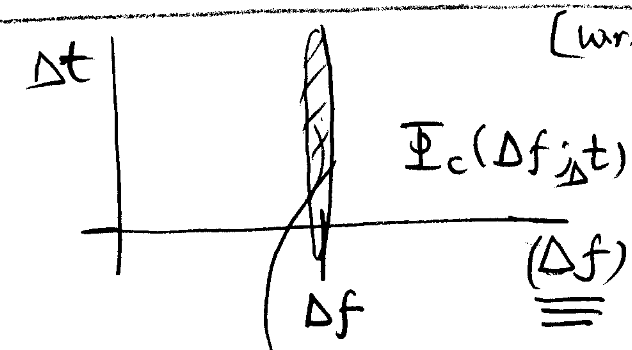
Now, spectral broadening due to mobility:

$$\boxed{S_c(\Delta f; \lambda) \equiv \int_{-\infty}^{+\infty} \Phi_c(\Delta f; \Delta t)\, e^{-j 2\pi \lambda (\Delta t)}\, d(\Delta t)}$$

(wr.t. t variable)

λ : Doppler frequency

$(\Delta t \longleftrightarrow \lambda)$



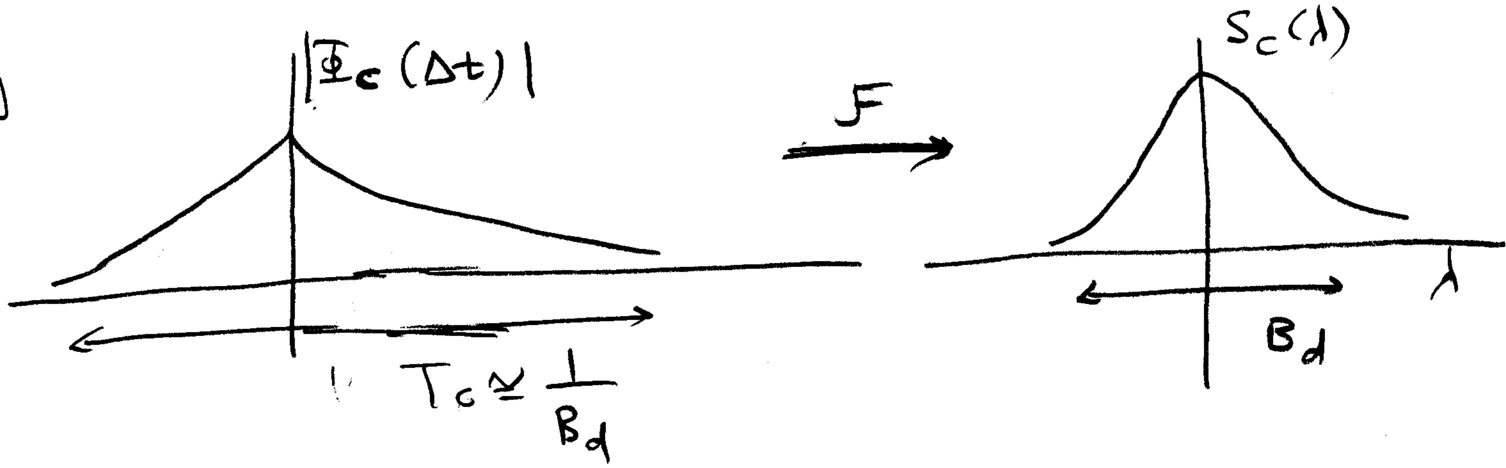
$\mathcal{F}$ transform along here gives the $S_c(\Delta f; \lambda)$

$$\Phi_c(\Delta t) \equiv \Phi_c(0, \Delta t)$$

$$S_c(\lambda) \equiv S_c(0; \lambda)$$

called the Doppler power spectrum of channel.

Ex)



* coherence time for a random channel is defined as the range of values for which the function $\Phi_c(\Delta t)$ is non-zero.

<u>More relationships</u>: Scattering function

$$\boxed{S(\tau;\lambda) \equiv \int_{-\infty}^{+\infty} \phi_c(\tau;\Delta t)\, e^{-j2\pi\lambda(\Delta t)}\, d(\Delta t)}$$

called the "<u>scattering function</u>" of the channel

Note that:

$$S(\tau;\lambda) = \int_{-\infty}^{+\infty} S_c(\Delta f;\lambda)\, e^{j2\pi\tau\Delta f}\, d(\Delta f).$$

i.e. $S(\tau;\lambda) \xrightarrow{\mathcal{F}} S_c(\Delta f;\lambda)$

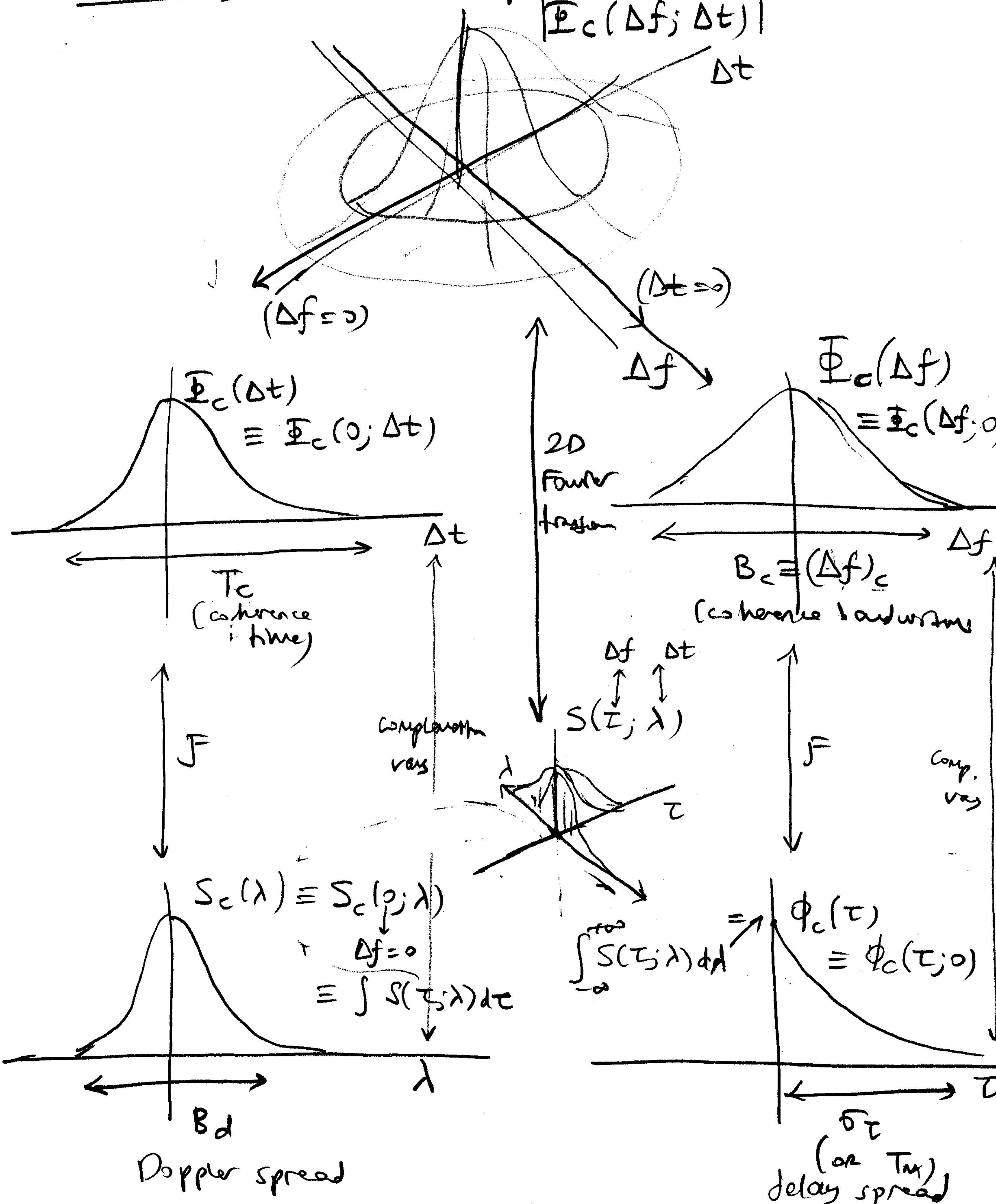
$(\tau \longleftrightarrow \Delta f)$

complementary variables

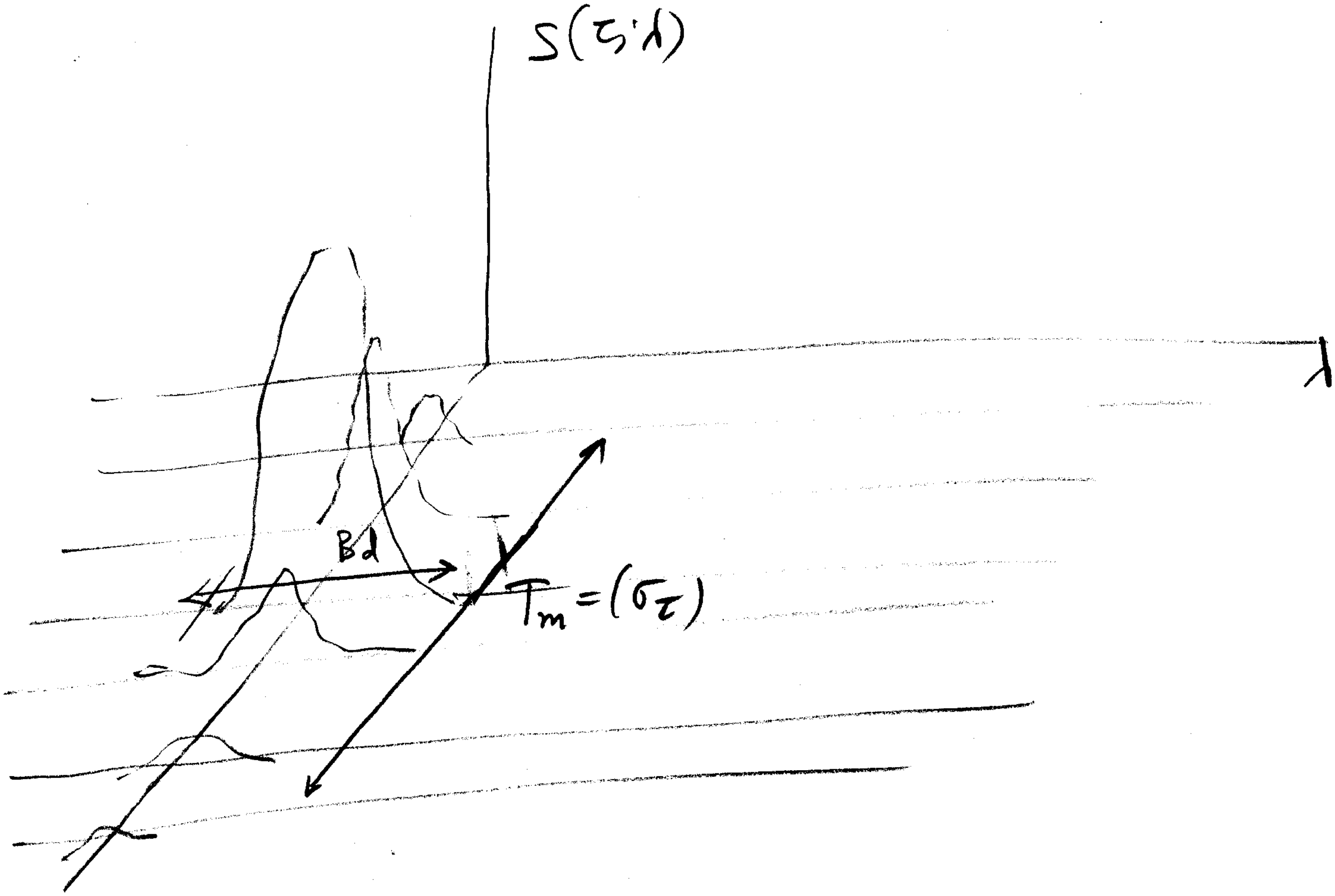
~~As a result: (substitute for $\phi_c(\tau;\Delta t)$ in terms of $\Phi_c(\Delta f;\Delta t)$~~

$$\boxed{S(\tau;\lambda) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_c(\Delta f;\Delta t)\, e^{-j2\pi\lambda(\Delta t)} \cdot e^{j2\pi\tau(\Delta f)} \cdot d(\Delta t)\, d(\Delta f).}$$

Summary of relationships:



Example channel: scattering function $S(\tau;\lambda)$



— Let's go back to our 2-ray channel model.

(b) What is the coherence time of this channel?

$$h_b(\tau) = \alpha_1 e^{j\phi_1}\delta(\tau) + \alpha_2 e^{j\phi_2}\delta(\tau - d)$$ relabeled d to avoid confusion

$[\phi_1 \perp\!\!\!\perp \phi_2,\ \alpha_1 \perp\!\!\!\perp \alpha_2,\ \phi_i \sim U[0,2\pi),\ \alpha_i \sim \text{Rayleigh}(\sigma_i)]$

d : delay - deterministic

Answer: We cannot determine based only on above information. The above information describes only the multipath characteristic & the value of the amplitude of the signal but does <u>not</u> tell you how the amplitudes across time t are correlated.

– Hence we would need more information.

(• <u>A model channel with a given $\phi_c(\tau; \Delta t)$</u> :

<u>Example</u> :

$$\phi_c(\tau; \Delta t) = \begin{cases} \text{sinc}(W\Delta t) & 0 \le \tau \le 10\mu s \\ 0 & \text{else.} \end{cases}$$

where $W = 100$ Hz, (and $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.)

(a) Is this an indoor or outdoor channel?

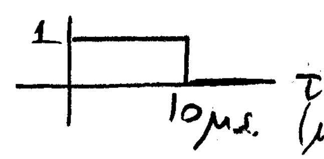
– <u>Answer</u>: outdoor. ($10\tilde{\mu s}$ delay spread)

(b) Sketch the <u>scattering function</u> of this channel.

<u>Answer</u>:

$$S(\tau; \lambda) \equiv \int_{-\infty}^{+\infty} \phi_c(\tau; \underbrace{\Delta t}_{\text{var. of integration}}) e^{-j2\pi\lambda(\Delta t)} \, d(\Delta t)$$

$$= \mathcal{F}_\lambda \{ \phi_c(\tau; \Delta t) \}$$



$$\phi_c(\tau; \Delta t) = \text{sinc}(W\Delta t) \cdot \text{rect}\left(\frac{t-5}{10}\right).$$

Δt: <u>seconds</u>.

$$\mathcal{F}_\lambda\{\phi(\tau; \underline{\Delta t})\} = \mathcal{F}_\lambda\left\{ \text{sinc}(W\Delta t) \cdot \text{rect}\left(\frac{\tau - 5}{10}\right) \right\}$$

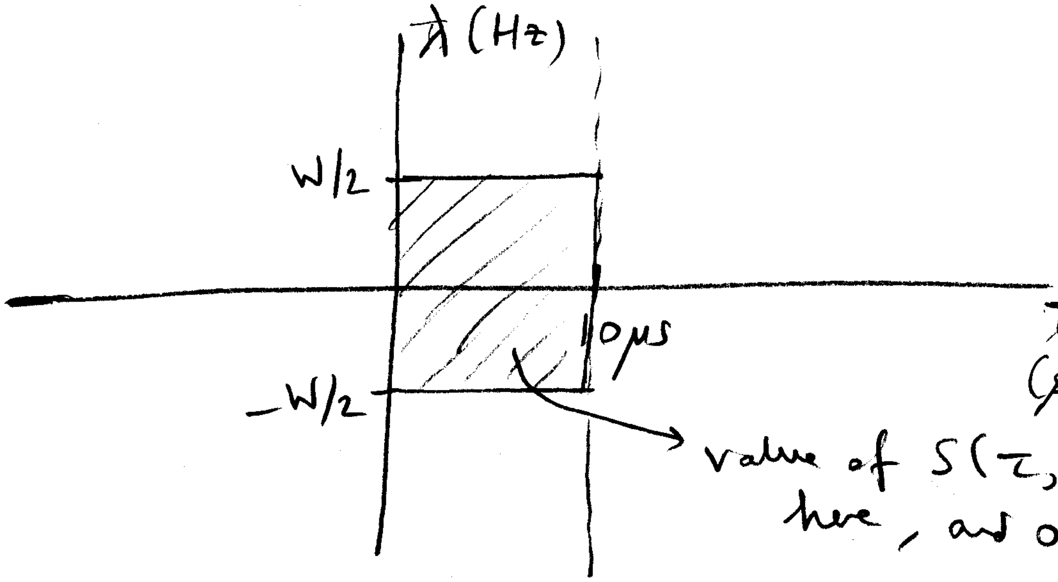
$$= \text{rect}\left(\frac{\tau - 5}{10}\right) \cdot \mathcal{F}_\lambda\{\text{sinc}(W\Delta t)\}$$

$$\boxed{S(\tau;\lambda) = \mathrm{rect}\left(\frac{\tau-5}{10}\right)\cdot\frac{1}{W}\,\mathrm{rect}\left(\frac{\lambda}{W}\right)}$$

[Recall: $\mathrm{rect}(t) \longleftrightarrow \mathrm{sinc}(f)$]

$\frac{\lambda}{W} = 1/2$

$\lambda = W/2$



value of $S(\tau;\lambda) = (1/W)$ here, and 0 elsewhere

(c) What is the d

i) average delay =

$$\boxed{\bar{\tau} = 5\,\mu s.}$$

ii) rms delay spread =

$$\overline{\tau^2} = \left[\frac{\int_0^{10}\tau^2 A_c(\tau)\,d\tau}{\int_0^{10}A_c(\tau)\,d\tau}\right] = \frac{\int_0^{10}\tau^2\,d\tau}{\int_0^{10}1\cdot d\tau} = \frac{\frac{\tau^3}{3}\Big|_0^{10}}{10} = \frac{1000}{3\cdot 10} = \frac{100}{3}$$

$$\bar{\tau}^2 = 25 \qquad \sigma_\tau^2 = \frac{100}{3} - \frac{75}{3} = \frac{25}{3} \qquad \boxed{\sigma_\tau = \frac{5}{\sqrt{3}}} \approx \boxed{2.87\,\mu s}$$

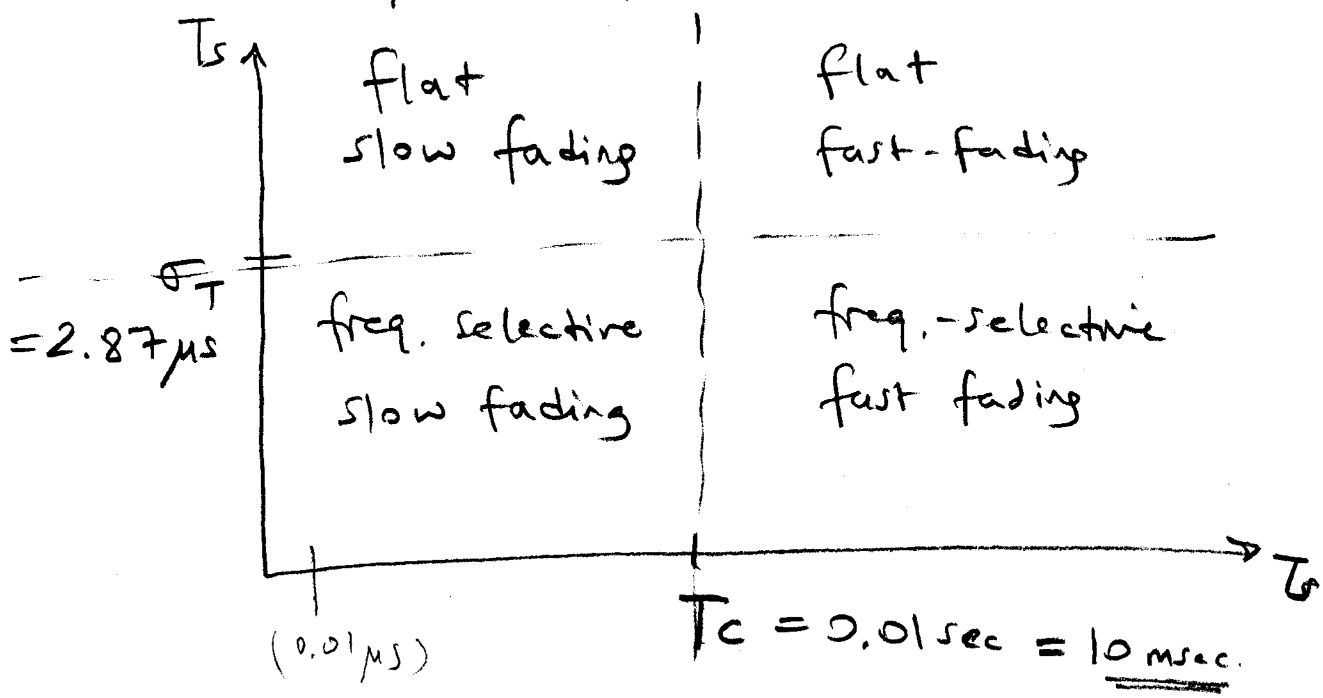
[or use $\frac{(b-a)^2}{12}$ for uniform distr.]

iii) Doppler spread: $\boxed{B_d = W \text{ "}100\text{" Hz.}}$

(nonzero over this range)

(iv) coherence time: $T_c = \frac{1}{B_d} = \frac{1}{100} = \boxed{0.01\text{ sec}}$

(d) Find the ranges for T_s when you get the 4 different fading / freq. selectivity conditions.



(e) Assume that $B_u = 100$ MHz

$(T_s = 0.01 \times 10^{-6}\,sec = 0.01\,\mu s.)$

Then: $T_s = 0.01\,\mu s \ll T_c \Rightarrow$ slow fading

and $T_s = 0.01\,\mu s \ll \sigma_T \Rightarrow$ freq. selective.

- possible scenarios: 1) wireless LANs: $5\,\mu s \cdot 3 \times 10^8$ m/sec $= 60$ meters extending over $\approx$ 100-150 meters outdoors; pedestrian motion or stationary user.

(f) Assume that $B_u = 1$ kHz

$(T_s = 1\,ms)$

$T_s = 1\,ms \lesssim 10\,msec. \Rightarrow \approx$ slow fading

$T_s = 1\,ms \gg \sigma_T = 2.87\,\mu s \Rightarrow$ flat channel

· Some acoustic channels
(not all)

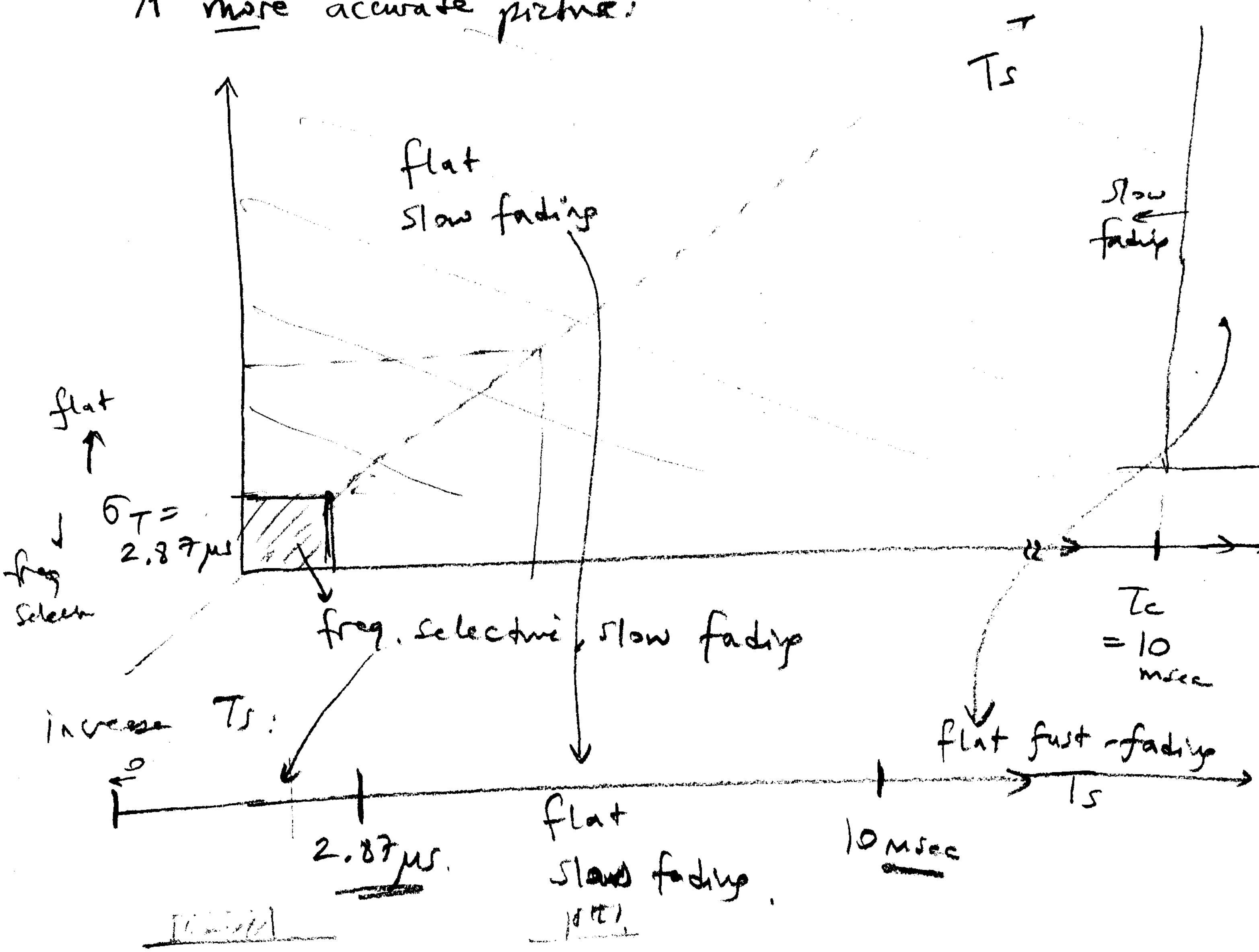
(g) Assume: $B_u = 1\ GHz$ (very high data rates)

$(T_s = 0.001\,\mu s)$

$T_s = 0.001\,\mu s \ll T_c \Rightarrow$ slow fading

$T_s = 0.001\,\mu s \ll 2.87\,\mu s \Rightarrow$ highly freq. selective.

A more accurate picture:



∴ This ~~signal~~ channel can experience only $\underline{\underline{3}}$ of the 4 regimes.

(h) In which regime do you expect Rayleigh to be a good model for the magnitude of complex envelope of received signal.

Answer: flat fast-fading

[The multipaths are not resolvable ⇒ they will cause destructive interference — small phase differences).