Lecture 3 – Robotics Preliminaries, part 2

RW

- **Rimless Wheel** (cont'd)

$\theta, \dot\theta$: 2 states, $x = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}$

but 1 degree-of-freedom

1 DOF, $q = [\theta]$

hybrid sys.: CT / DT

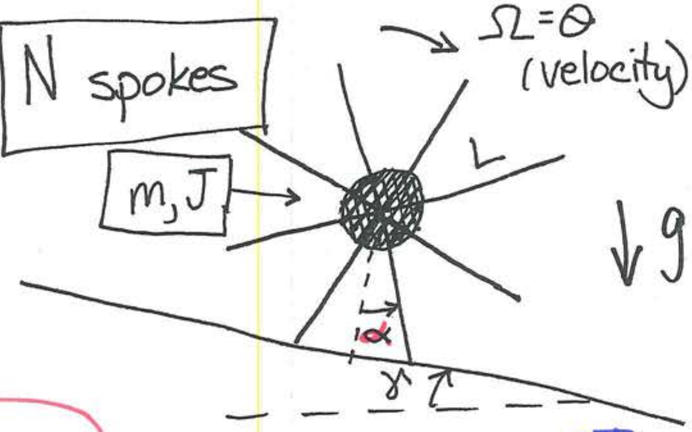
  - continuous dynamics: $\boxed{\ddot\theta = f(\theta, \dot\theta)}$
  - discrete jumps in velocity: $\boxed{\dot\theta^- \to \dot\theta^+}$
  - "return map": Given $\dot\theta_n^+$, what is $\dot\theta_{n+1}^+$ ← next steps
  - "basin of attraction": Given $\dot\theta_n^+$, what is $\dot\theta_\infty^+$

- **Continuous-time (CT) dynamics**
  - Lagrangian method ← RW: 1 eq. of motion (EOM), b/c we have 1 DOF (θ). (n eqns for n DOF's).

- **Underactuation** (vs controllability)

Cannot independently & instantaneously set all n accelerations ($\ddot q$) for the n DOF's, with control u.

Rimless Wheel dynamics



$2\alpha = \frac{2\pi}{N}$

$\boxed{\alpha = \frac{\pi}{N}}$

DT event

- **Impacts:**

$\boxed{\Omega \equiv \dot{\theta}}$

$$\Omega_n^{+} = \eta\, \Omega_n^{-}$$

post-impact (Ω_n^{+}), pre-impact (Ω_n^{-})

$$\eta = \frac{J + mL^2\cos(2\alpha)}{J + mL^2}$$

CT dynamics:

- **Conservation of Energy:**

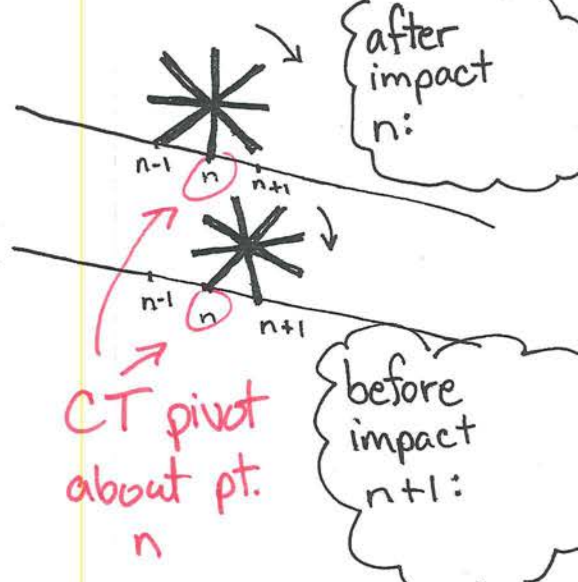
$$\boxed{E_{n+1}^{-} = E_n^{+}}$$

$$(K.E.)_{n+1}^{-} + (P.E.)_{n+1}^{-} = (K.E.)_n^{+} + (P.E.)_n$$

K.E.: Kinetic energy = **T**

P.E.: potential energy = **V**

$$K.E.:\ T = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} m (\underbrace{L\Omega}_{\text{vel. of C.O.M (center of mass)}})^2 + \tfrac{1}{2}\underbrace{J}_{\text{wrt com}}\Omega^2$$

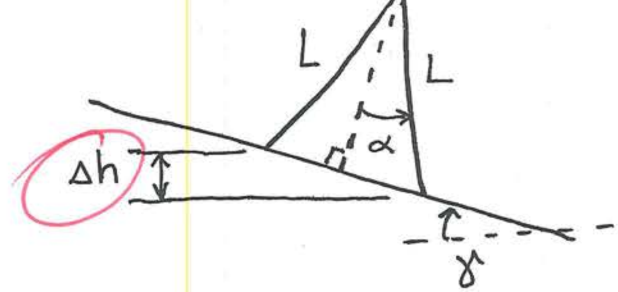


$-\Omega < 0$, up

$\Omega > 0$, downhill

Height, h, of Center of Mass (COM)

"P.E." : $V = mgh + \text{constant}$



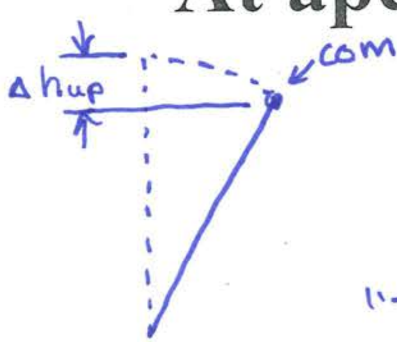
- **At impact states:**

$\Delta V = mg\Delta h$

$\Delta h = \pm 2L \sin\alpha \sin\gamma$

uphill ($+$), downhill ($-$)

- **At apex states:**

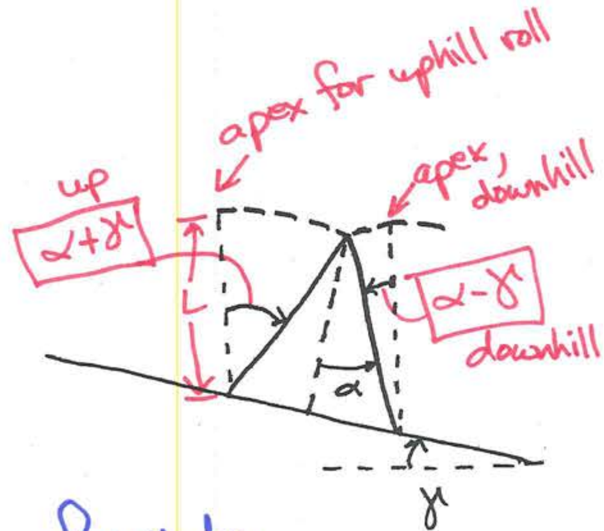


$$\Delta h_{crit,up} = L(1-\cos(\alpha+\gamma))$$

up

$$\Delta h_{crit,dn} = L(1-\cos(\alpha-\gamma))$$

dn



"Teeters" when $\Omega_{crit,up} < \Omega < \Omega_{crit,dn}$

- **Critical values?** ← Plug in Δh_{crit} ...

$$\Omega_{crit,up} = -\sqrt{\frac{2mgL(1-\cos(\alpha+\gamma))}{(J+mL^2)}}$$

uphill

$$\Omega_{crit,dn} = +\sqrt{\frac{2mgL(1-\cos(\alpha-\gamma))}{(J+mL^2)}}$$

Solve for K.E. $+ \Delta$P.E.$_{apex} = 0$

Impact-to-Impact dynamics – 3 Cases

$\Omega > \Omega_{crit,dn}$ $\begin{cases} ① \text{ CT}: \frac{1}{2}(J+mL^2)(\Omega^-_{n+1})^2 = \frac{1}{2}(J+mL^2)(\Omega^+_n)^2 - mg\Delta h \\ ② \text{ DT}: \text{mult. by } \eta \end{cases}$

$\Omega^+_{n+1} = f(\Omega^+_n)$

A: Downhill

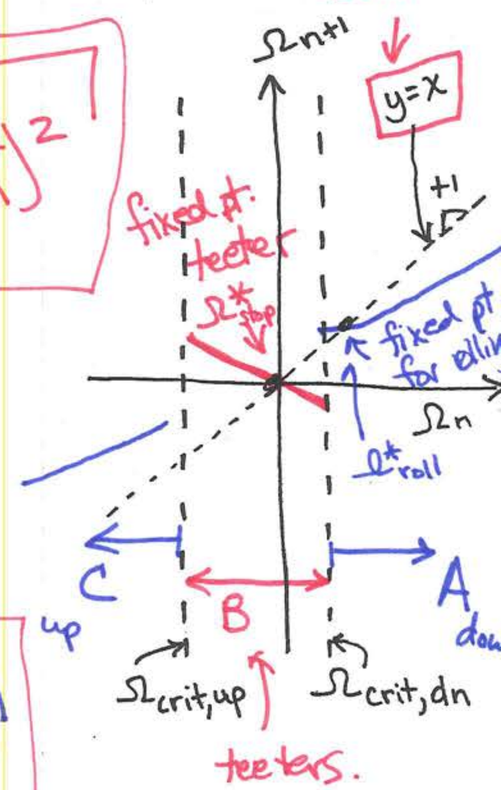
$$\boxed{\Omega^+_{n+1} = \eta\sqrt{\frac{4mgL\sin\alpha\sin\gamma}{J+mL^2} + (\Omega^+_n)^2}}$$

B: "Teeters"

$$\boxed{\Omega^+_{n+1} = \eta\,\Omega^+_n}$$

C: Uphill

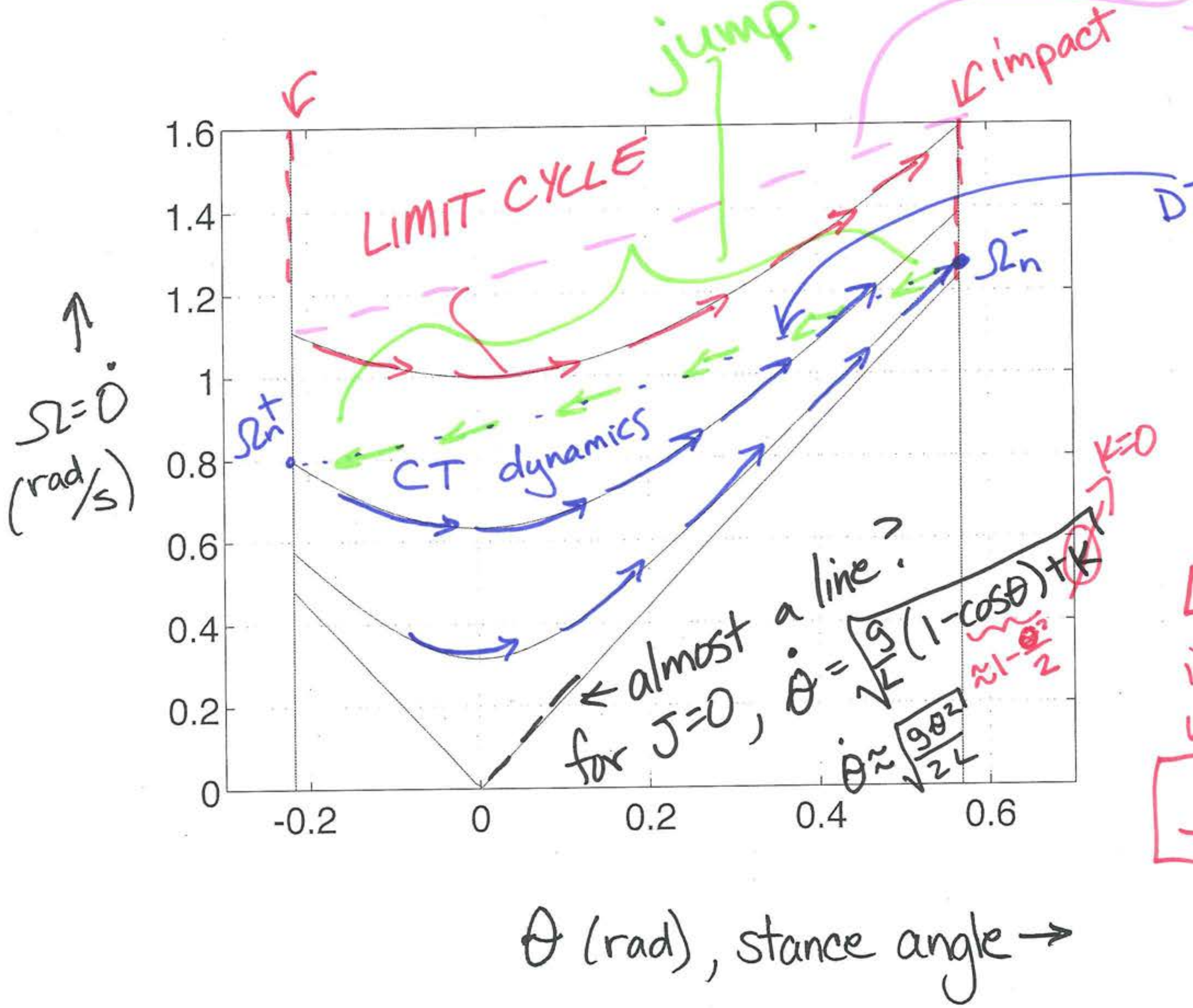
$$\Omega^+_{n+1} = \boxed{-\eta\sqrt{\frac{-4mgL\sin\alpha\sin\gamma}{J+mL^2} + (\Omega^+_n)^2}}$$



$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ ← 2 states, positions & velocities.

"Flow" in 2D CT state space:

jump goes to start of same CT traj.



DT "jump" in state space, to start the new stance phase.

LIMIT CYCLE is the case where

$$\boxed{\Omega_{n+1}^+ = \Omega_n^+}$$

$= \Omega^*$

fixed point

Poincaré section

- **Return map:** "What will the state be when we RETURN to a given constraint?" The mapping from Ω_n^+ to Ω_{n+1}^+

- **Fixed point:** A value that maps to itself on the return map.

When $\Omega_{n+1}^+ = \Omega_n^+ = \Omega^*$, e.g. $\Omega^* = 0$ (standing still)

- **Cobweb plot:** Graphical way to predict step-to-step transitions on a 1D return map. See Matlab & HW1.

- **Basin of attraction:** A region in state space that will evolve toward a particular "attractor", such as a fixed point.

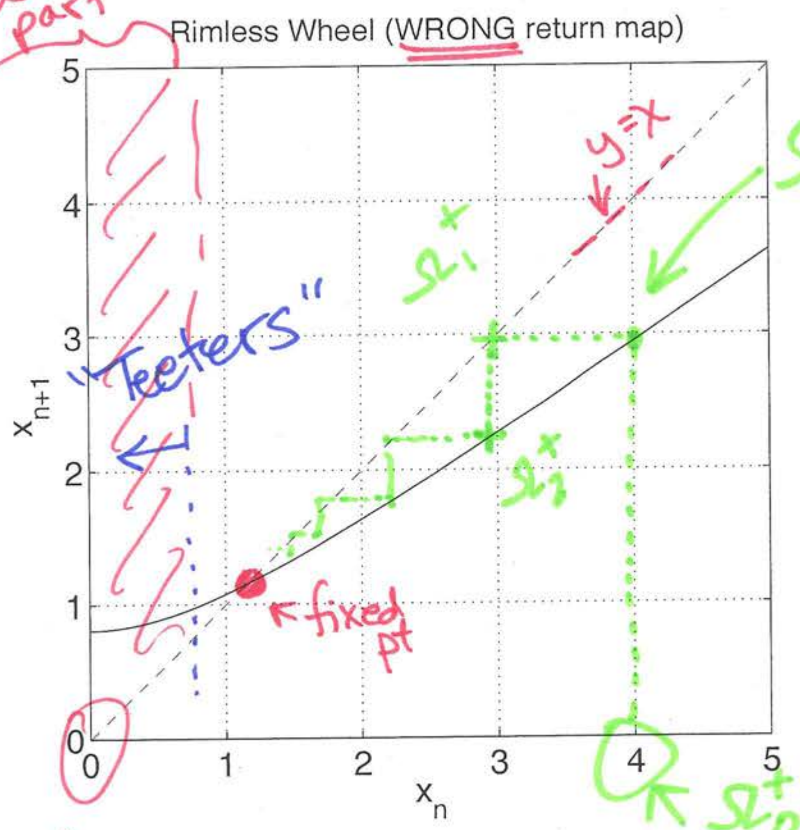
See next pages!

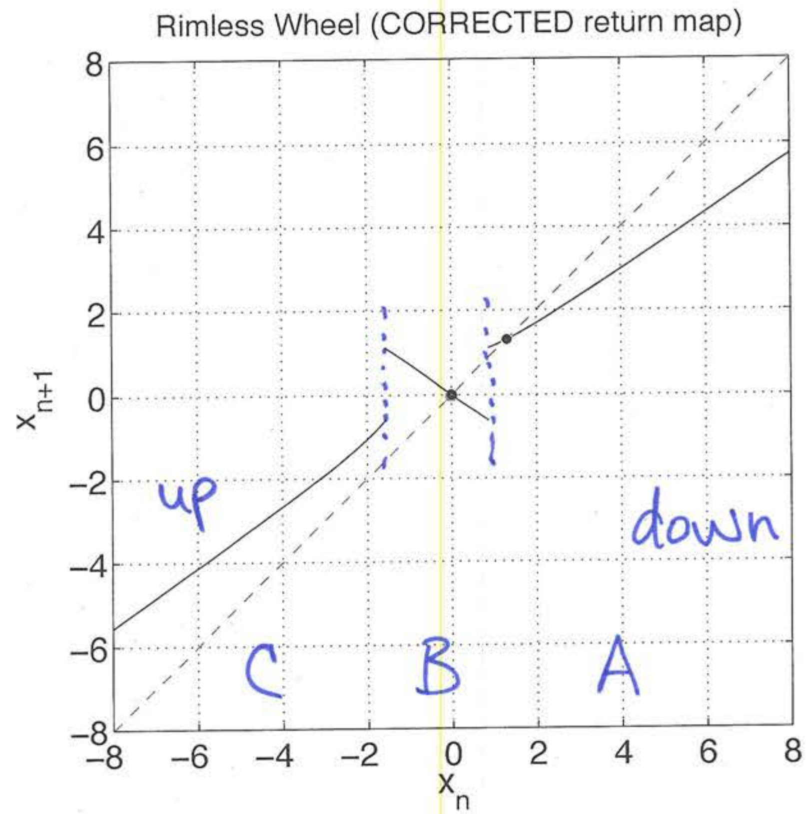
See MATLAB example code (website)

MATLAB:
rw_demo_corrected.m

"Cobweb"

Bad part



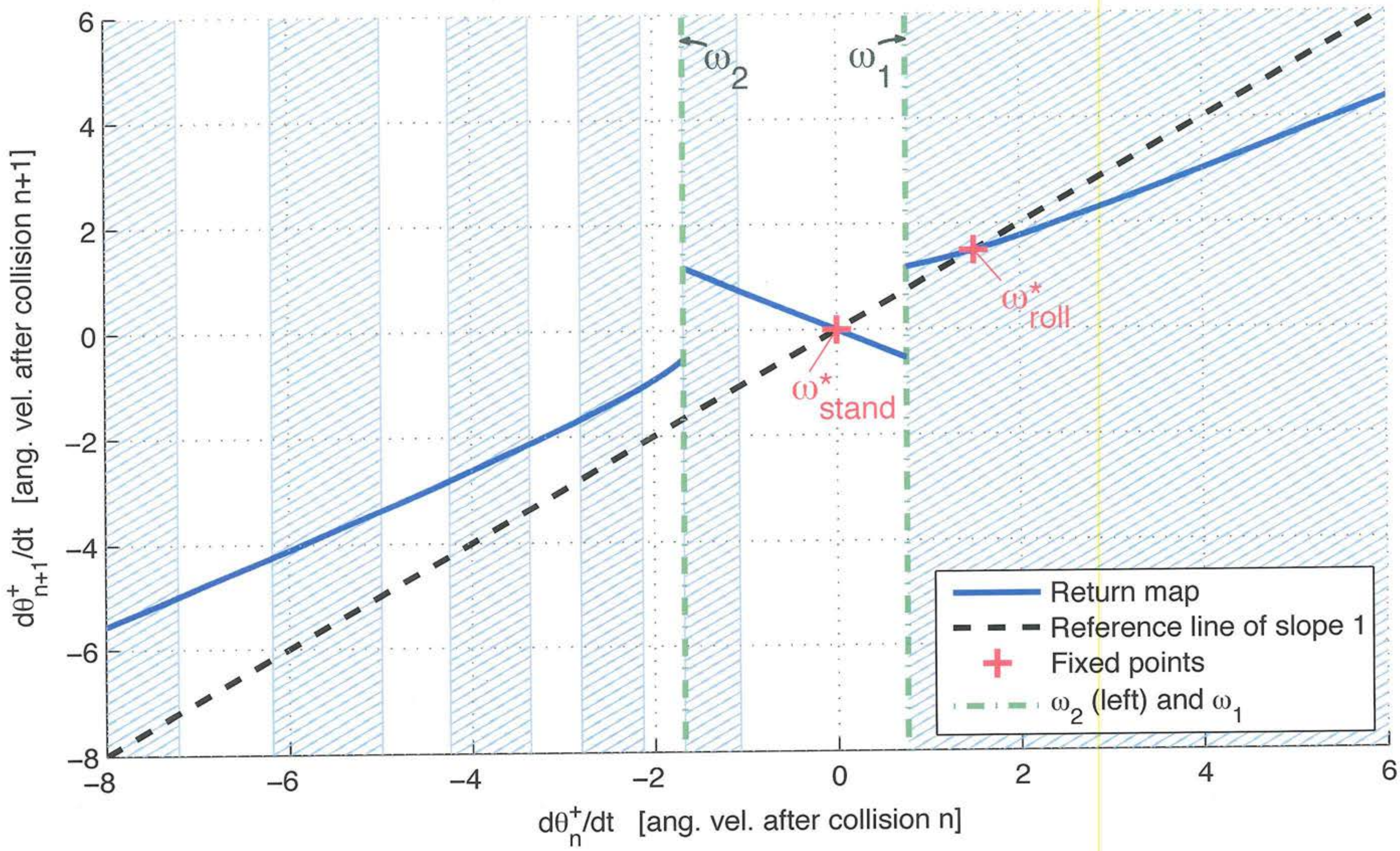


energy balance
ONLY...

Wrong
intuitively
(when $\Omega_n^+ = 0$, $\Omega_{n+1}^+ = 0$, ...)

Refer to for HW 1.

Rimless Wheel: Basins of Attraction



CT Dynamics – can be written in the form:

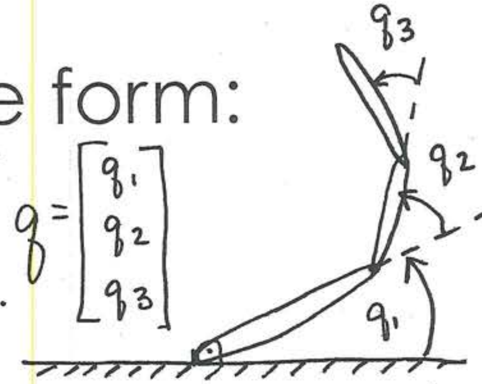
vs state vector, w/pos + vel:

$$x = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$\boxed{\ddot{q} = f_1(q,\dot{q},t) + f_2(q,\dot{q},t)\, u}$$

← u: control inputs.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



manipulator example.

q holds only DOF's (positions, <u>NOT</u> velocities).

$\boxed{q: \text{"generalized coordinates"}}$

- **First, how do get get f1 and f2?**

Lagrangian method.

- **Second, when is the system <u>underactuated</u>?**

When $\boxed{\text{rank}(f_2) < n}$ ← n DOF in q.

← Why? B/c for any "$\ddot{q}_{des}$, $u = \underbrace{f_2^{-1}}_{\text{inverse}}(\ddot{q}_{des} - f_1)$"

- **Third, when is it <u>controllable</u>?**

If we can get from one location to another in state space, via some path. $x_a \rightarrow x_b$

Lagrangian method

1. Construct the "Lagrangian":

Kinetic "co-energy"

$$\mathcal{L} = T^* - V$$

"K.E." $\boxed{\sum \frac{1}{2}mv^2}$ "P.E." $\boxed{mgh}$ & $\boxed{\frac{1}{2}k\Delta x^2}$

2. Left-Hand Side (LHS) is:

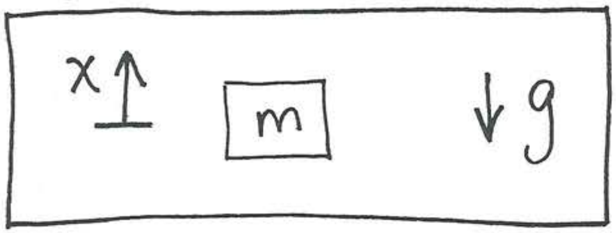
$$\left.\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) - \frac{\partial \mathcal{L}}{\partial q_n}\right\} \leftarrow \text{conservative terms}$$

3. Right-Hand Side (RHS) is:

Ξ_n } generalized non-conserv. terms:
e.g. { u_n: inputs $\leftarrow$
{ "$-b\dot{x}$" $\leftarrow$ damping

Example 1 – Dropping a free mass

1 DOF { "x"



$$T^* = \frac{1}{2} m \dot{x}^2$$

$$V = mgx$$

$$\mathcal{L} = T^* - V$$

1 EOM

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \underbrace{m\ddot{x} + mg}_{\text{LHS}} = \underbrace{0}_{\text{RHS}}$$

Constant, negative accel. →

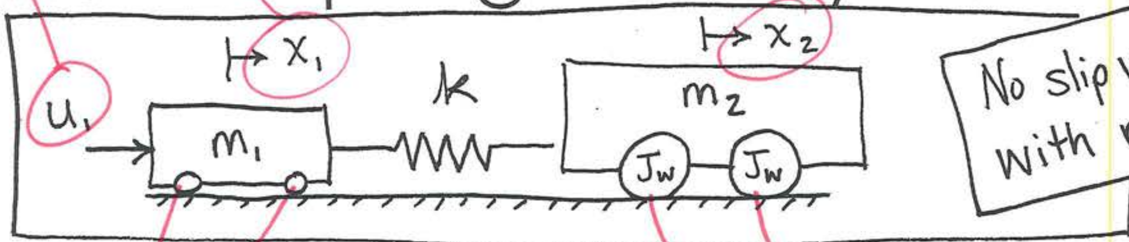
$$\boxed{\ddot{x} = -g}$$

Example 2 – Spring-mass system

1 input

2 DOF'S

$$q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



No slip wheels with radius R.

$J=0$ for m_1 wheels

$J \neq 0$

2 wheels

$$T^* = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2}(2J_w)\dot{\theta}^2$$

$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \left(\tfrac{1}{2} m_2 + \frac{J_w}{R^2}\right)\dot{x}_2^2$$

no slip, so

$x_2 = R\theta$

$$\therefore \boxed{\dot{\theta} = \frac{\dot{x}_2}{R}}$$

$$V = \tfrac{1}{2} k (x_1 - x_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = \boxed{u_1 = m_1 \ddot{x}_1 + k(x_1 - x_2)}$$

for $x_2 \Rightarrow$

$$\boxed{0 = \left(m_2 + 2\frac{J_w}{R^2}\right)\ddot{x}_2 - k(x_1 - x_2)}$$

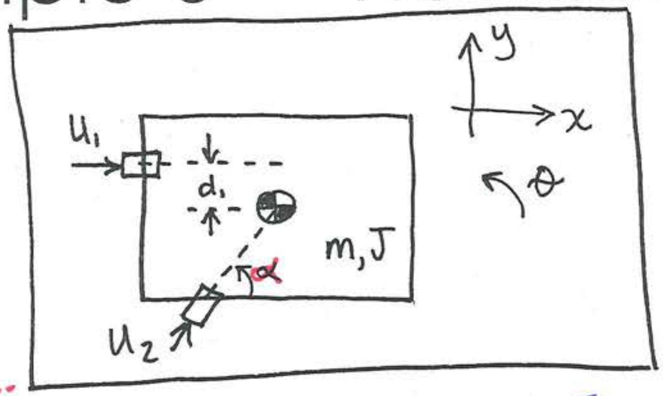
rank$(f_2) = 1$

(under- act

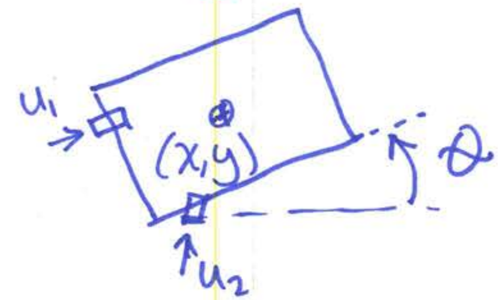
$$\therefore \text{"}\ddot{q} = f_1 + f_2 u\text{" yields: } f_1 = \begin{bmatrix} \dfrac{-k(x_1 - x_2)}{m_1} \\ \dfrac{+k(x_1 - x_2)}{\left(m_2 + 2\frac{J_w}{R^2}\right)} \end{bmatrix}, \; f_2 = \begin{bmatrix} \frac{1}{m_1} \\ 0 \end{bmatrix}$$

Example 3 – "Hovercraft" – ...floating on a constant-height, frictionless table.

u_1 & u_2 are <u>fixed</u> on the body...



...so, if body rotates,



$$q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \leftarrow 3 \text{ DOF's}$$

$$T^* = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \dot{y}^2 + \tfrac{1}{2} J \dot{\theta}^2$$

$$V = 0$$

$\text{rank}(f_2) = 2$ (2 actuators)

3 eqns:

$$\boxed{\begin{aligned} m\ddot{x} &= u_1 \cos\theta + u_2 \cos(\theta + \alpha) \\ m\ddot{y} &= u_1 \sin\theta + u_2 \sin(\theta + \alpha) \\ J\ddot{\theta} &= -u_1 d_1 \end{aligned}}$$

$$f_2 = \begin{bmatrix} \cos\theta, & \cos(\theta+\alpha) \\ \sin\theta, & \sin(\theta+\alpha) \\ -d, & 0 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$