Cart-pole system: Equations of motion

Nonlinear Dynamics

This document provides a derivation of the equations of motion (EOM) for the cart-pole system. The (true) nonlinear dynamic equations are derived first, using a Lagrangian approach; then the system is linearized about the upright equilibrium ("inverted pendulum") position. Figure 1 shows the system. There are two degrees of freedom: the position of the cart, x, and the angle of the pendulum, θ. The system is underactuated, since there is only one actuation: a force, F_x, applied on the cart.
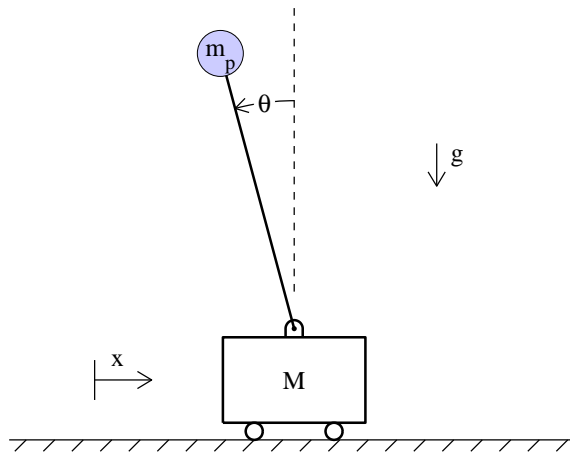

Figure 1: Cart-pole system.

To begin, we will derive the simple relationships giving the position and velocity of the pendulum:

$$x_p = -L \sin \theta \tag{1}$$

$$\dot{x}_p = -L \cos \theta \dot{\theta} + \dot{x} \tag{2}$$

$$y_p = L \cos \theta \tag{3}$$

$$\dot{y}_p = -L \sin \theta \dot{\theta} \tag{4}$$

The "Lagrangian" for a dynamic system is defined as:

$$\mathcal{L} = T^* - V \tag{5}$$

where T^* is the *kinetic energy* and V is the *potential energy*. For the cart-pole system:

$$
\begin{aligned}
T^* &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_p \left(\dot{x}_p^2 + \dot{y}_p^2 \right) \\
&= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_p \left(\left(L^2 \cos^2 \theta \dot{\theta}^2 - 2L \cos \theta \dot{\theta} \dot{x} + \dot{x}^2 \right) + \left(L^2 \sin^2 \theta \dot{\theta}^2 \right) \right) \\
&= \frac{1}{2} \left(M + m_p \right) \dot{x}^2 + \frac{1}{2} m_p \left(L^2 \cos^2 \theta \dot{\theta}^2 + L^2 \sin^2 \theta \dot{\theta}^2 \right) - m_p L \cos \theta \dot{\theta} \dot{x} \\
&= \frac{1}{2} \left(M + m_p \right) \dot{x}^2 + \frac{1}{2} m_p L^2 \dot{\theta}^2 - m_p L \cos \theta \dot{\theta} \dot{x}
\end{aligned}
\tag{6}
$$

And the potential energy is:

$$V = m_p g y_p$$
$$= m_p g L \cos \theta \tag{7}$$

So the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} \left(M + m_p \right) \dot{x}_p^2 + \frac{1}{2} m_p L^2 \dot{\theta}^2 - m_p L \cos \theta \dot{\theta} \dot{x} - m g L \cos \theta \tag{8}$$

Next, for each generalized coordinate, q_n, and its associated actuating force (if any), Ξ_n, we can derive an equation of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{d\mathcal{L}}{q_n} = \Xi_n \tag{9}$$

For the x coordinate of the cart:

$$\Xi_n = F_x \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(M + m_p \right) \dot{x} - m_p L \cos \theta \dot{\theta} \tag{11}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(M + m_p \right) \ddot{x}_p + m_p L \sin \theta \dot{\theta}^2 - m_p L \cos \theta \ddot{\theta} \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \tag{13}$$

Combining these expressions as required by Equation 8, we obtain one equation of motion:

$$F_x = \left(M + m_p \right) \ddot{x} + m_p L \sin \theta \dot{\theta}^2 - m_p L cos \theta \dot{\theta} \tag{14}$$

For the (unactuated) angle of the pendulum, θ_p,

$$\Xi_\theta = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_p L^2 \dot{\theta} - m_p L \cos \theta \dot{x}_p \tag{16}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_p L^2 \ddot{\theta} - m_p L \cos \theta \ddot{x} + m_p L \sin \theta \dot{\theta} \dot{x}_p \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m_p L \sin \theta \dot{\theta} \dot{x} + m_p g L \sin \theta \tag{18}$$

Again using Equation 8, we get the second equation of motion:

$$0 = m_p L^2 \ddot{\theta} - m_p L \cos \theta \ddot{x} + m_p L \sin \theta \dot{\theta} \dot{x}_p - m_p L \sin \theta \dot{\theta} \dot{x} - m_p g L \sin \theta$$
$$= m_p L^2 \ddot{\theta} - m_p L \cos \theta \ddot{x} - m_p g L \sin \theta \tag{19}$$

Linearization

Assuming the pendulum remains near $\theta = 0$, we can linearize the equations of motion given in 14 and 19. For the linearization, we assume:

$$\sin \theta \approx \theta$$
$$\cos \theta \approx 1$$
$$\dot{\theta}^2 \approx 0$$

Equation 14 can then be approximated as:

$$F_x = (M + m_p)\ddot{x} - m_p L \ddot{\theta} \tag{20}$$

and equation 19 becomes:

$$0 = -m_p L \ddot{x} + m_p L^2 \ddot{\theta} - m_p g L \theta \tag{21}$$

Decoupling the equations of motion

It is trivial to rewrite the dynamics represented by 20 and 21 to give the following two (explicit) expressions for the accelerations of the state variables of the *linearized* system:

$$\ddot{\theta} = \left(\frac{M + m_p}{M}\right)\frac{g}{L}\theta + \left(\frac{1}{ML}\right)F_x \tag{22}$$

$$\ddot{x} = \left(\frac{m_p}{M}\right)g\theta + \left(\frac{1}{M}\right)F_x \tag{23}$$

After a bit of mathematical gymnastics, we can also rewrite the two nonlinear equations of motion (14 and 19) to express each acceleration explicitly:

$$\ddot{\theta} = \frac{-sin\theta m_p L \cos\theta \dot{\theta}^2 + \sin\theta g M + \sin\theta m_p g + \cos\theta F_x}{L\left(-m_p \cos^2\theta + M + m_p\right)} \\ = \frac{-m_p L \sin\theta \cos\theta \dot{\theta}^2 + (M + m_p)g\sin\theta + \cos\theta F_x}{(M + m_p(1 - \cos^2\theta))L} \tag{24}$$

$$\ddot{x} = \frac{-sin\theta m_p L \dot{\theta}^2 + \sin\theta m_p \cos\theta g + F_x}{-m_p\cos^2\theta + M + m_p} \\ = \frac{-m_p L \sin\theta \dot{\theta}^2 + m_p g \sin\theta \cos\theta + F_x}{M + m_p(1 - \cos^2\theta)} \tag{25}$$