Homework 1 (due 1/20)

1.1 2D Robot Platform. In Figure 1, the large rectangle represents a robot platform that can move (friction free) in the plane of the paper. (For example, imagine the platform floats on an air hockey table.) The system has 3 degrees of freedom (DOFs): x, y, and θ. It also has 3 actuators. Each actuator is rigidly mounted at a particular location and angle on the perimeter of the platform and can provide thrust (+ or - direction) along its axis.

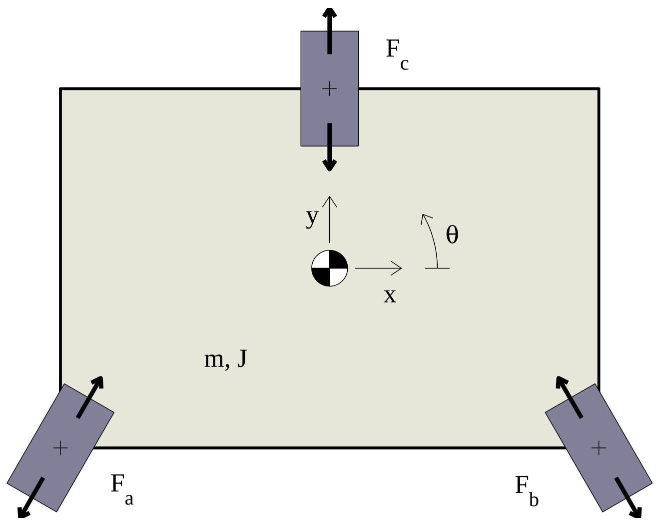

Figure 1: 2D robot platform.

a) Assume the 3 thrust actuators are at the following locations (in meters) and angles:

F_a: $x = -0.75, y = -0.5, \theta = 60°$
F_b: $x = 0.75, y = -0.5, \theta = 120°$
F_c: $x = 0, y = 0.5, \theta = 90°$

We want to determine if this system is *fully actuated* or *underactuated*. That is, we aim to find combinations of thrust that decouple the motions, so we can move in each of the 3 DOFs independently. Recall the robot manipulator equation:

$$\ddot{q} = \mathbf{f_1}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{f_2}(\mathbf{q}, \dot{\mathbf{q}}, t)\mathbf{u}$$

(i) Given the current configuration, what is $\mathbf{f_2}$? (i.e., What are the values in this 3x3 matrix?)
(ii) What is the rank of $\mathbf{f_2}$? Is the system fully- or underactuated?
(iii) Is the system controllable? (Does a path exist to get to any desired location and orientation?) Is the "controllability matrix" described in class a valid way to determine this? Why or why not?

b) You are asked to reposition the actuators on the platform.
(i) Find a set of locations and angles (x, y, θ) for each of the 3 actuators (along the perimeter of the platform) for which the system is fully actuated.
(ii) Calculate $\mathbf{f_2}$ (which should now have "full rank").

1.2 Nonlinear, underactuated cart-pole system. The main purpose of this problem is to ensure everyone has access to run and a basic ability to use MATLAB. You will be running and modifying code to simulate the dynamics of the classic "cart-pole" system shown in Figure 2.

a) Download the three m-files for HW1 from the homework page of the course website. Run the script called "`cartpole_hw1a.m`" and include in your homework a print-out of the MATLAB figure produced. This code uses "`cartpole_linear.m`" and ode45 to simulate the *linearized* dynamics of system, stabilized using linear quadratic regulator (LQR) control.

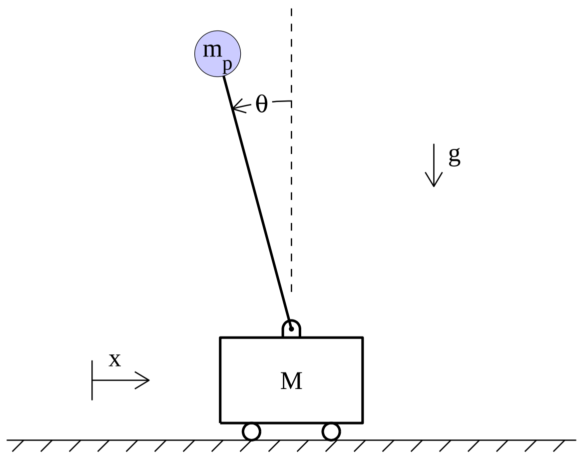

Figure 2: Cart-pole system.

b) The nonlinear equations of motion for the system are:

$$(M + m_p)\,\ddot{x} - m_p L \cos\theta\ddot{\theta} + m_p L \sin\theta\dot{\theta}^2 = F_x \tag{1}$$

$$-m_p L \cos\theta\ddot{x} + m_p L^2\ddot{\theta} - m_p g L \sin\theta = 0 \tag{2}$$

These equations can be decoupled to give explicit expressions for the accelerations, $\ddot{x}$ and $\ddot{\theta}$:

$$\ddot{\theta} = \frac{-m_p L \sin\theta\cos\theta\dot{\theta}^2 + (M + m_p)\,g\sin\theta + \cos\theta F_x}{(M + m_p\,(1 - \cos^2\theta))\,L} \tag{3}$$

$$\ddot{x} = \frac{-m_p L \sin\theta\dot{\theta}^2 + m_p g \sin\theta\cos\theta + F_x}{M + m_p\,(1 - \cos^2\theta)} \tag{4}$$

Starting with a copy of "`cartpole_linear.m`", create a new m-file called "`cartpole_nonlinear.m`" that simulates the true, nonlinear dynamics in Equations 3 and 4. Compare the linearized and nonlinear responses by running "`cartpole_hw1b`" in MATLAB, and turn in print-outs of the new plots that are automatically generated.

c) Finally, test your code with various initial conditions $x = 0$, $\dot{x} = 0$, $\theta = \theta_o$, $\dot{\theta} = 0$, and find the largest value of θ_0 for which the nonlinear system will successfully balance at the top. (You don't need to be more precise than about 1/2 a degree or so...) This initial condition will be near the *separatrix* of the *basin of attraction* of the controlled system.)

1.3 Brockett Integrator. Consider an underactuated, 3 DOF system with control inputs u and v that obeys the following equations of motion:

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = yu - xv = y\dot{x} - x\dot{y}$$

This is a famous dynamic system, often referred to as the "Brockett Integrator" and described in more detail in [**?**], and it is controllable. Your task in this problem is to study the dynamics to develop a path-planning strategy to travel from any initial (x_o, y_o, z_o) location to any final location: (x_f, y_f, z_f). We can move in any (x, y) trajectory we wish; the issue/difficultly is in controlling z. Let's begin by examining some particular trajectories.

a) Start at the (x, y, z) point $(10, 0, 0)$. Let $u = 0$ and $v = 1$ (so you move purely in the y direction) until you arrive at ($x = 10$, $y = 5$). Integrate the expression for $\dot{z}$ over this motion to find the new value of z. What would the new value of z be if we had traveled the same trajectory with $u = 0$ and $v = 10$?

b) Start again at $(10, 0, 0)$ and trace out a square by moving to ($x = 10, y = 5$), then ($x = 5$, $y = 5$), then ($x = 5, y = 0$) and finally ($x = 10, y = 0$). as shown at left in Figure 3. (Assume u or v is equal to either 0 or 1, as appropriate, to trace this shape.) What is the value of z at each "corner" of the path?

c) Now, start at $(10, 10, 0)$ and trace the same size square, moving around the perimeter once again in a counter-clockwise direction, as illustrated on the right half of Figure 3. What is the values of z at each corner of this new path?

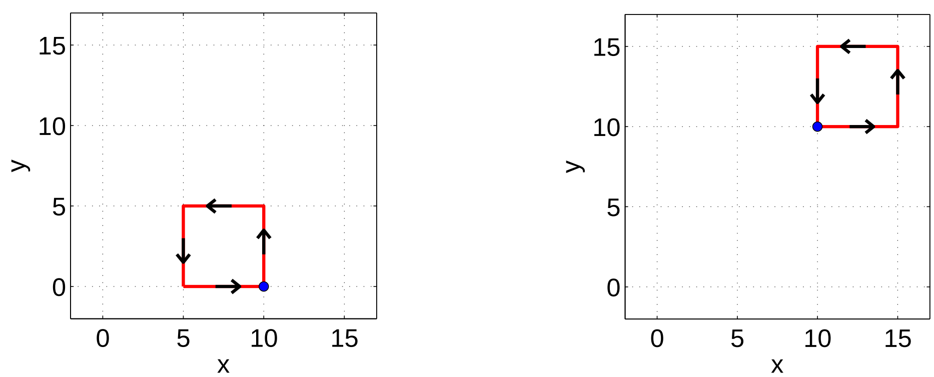

Figure 3: The Brockett integrator.

d) Extra credit.
i) Find a closed path that begins and ends at $(x = 0, y = 0)$ and achieves $\Delta z = -9$. ($\Delta z = z_{final} - z_{initial}$)
ii) Find the/a optimal solution to i) that minimizes the distance of the (x, y) path taken.
iii) Find any path that begins at $(2, -3, 0)$ and ends at $(3, 1, 9)$.