Homework 4 (due 2/24)

4.1 Lagrangian equations of motion: Two-link pendulum.

a) For the system shown in Figure 1, write expressions for T^* (the kinetic energy), for V (the potential energy) and for the Lagrangian: $\mathscr{L} = T^* - V$. (Hint: write expressions for the position of the mass [e.g., for x_m and y_m] and differentiate to find its velocity.)

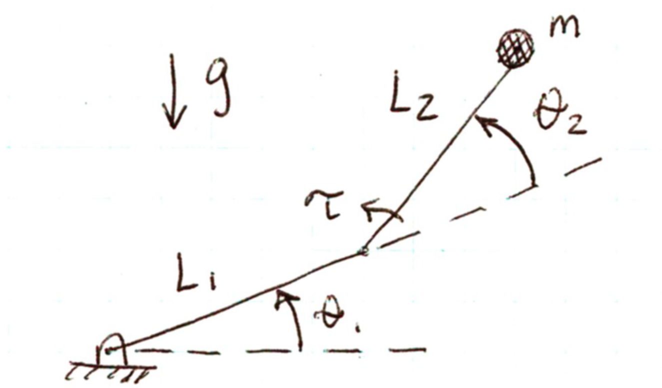

Figure 1: Double-link pendulum with point mass and torque (τ) at elbow.

b) Find the equations of motion (EOMs) for the system. For each degree of freedom ($q_1 = \theta_1$, $q_2 = \theta_2$), $\Xi_i = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i}$. For this underactuated system, $\Xi_i = 0$ and the second actuation is $\Xi_i = \tau$. (τ appears in only the second EOM because we have used the *relative* angle at the elbow as the second degree of freedom.)

position and balance it about the vertical. The equa-

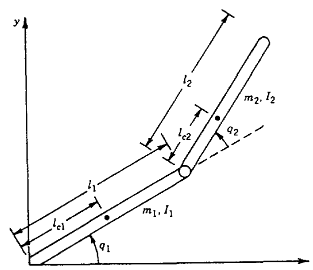

Figure 3: The Acrobot

tions of motion for the Acrobot are

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \quad (51)$$
$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau_2 \quad (52)$$

where

$$\begin{aligned}
m_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos(q_2)) + I_1 + I_2 \\
m_{22} &= m_2\ell_{c2}^2 + I_2 \\
m_{12} &= m_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos(q_2)) + I_2 \\
h_1 &= -m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2^2 - 2m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2\dot{q}_1 \\
h_2 &= m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_1^2 \\
\phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos(q_1) + m_2\ell_{c2}g\cos(q_1 + q_2) \\
\phi_2 &= m_2\ell_{c2}g\cos(q_1 + q_2)
\end{aligned}$$

The simulations to follow were written in Simnon using the parameters in Table 1 below.

m_1	m_2	ℓ_1	ℓ_2	ℓ_{c_1}	ℓ_{c_2}	I_1	I_2	g
1	1	1	1	0.5	0.5	0.2	1.0	9.8

Table 1: Parameters of the Simulated Acrobot

Figure 2: Equations of motion for the acrobot, from [1] (p.381)

c) This system is really just an acrobot. Show that your solution in part **a)** is identical to the equations of motion of an acrobot (described, for instance, in Spong's PFL paper [1]) if we use $m_1 = 0$, $L_2 = L_{c2}$, and $I_1 = I_2 = 0$. These equations of motion are shown in Figure 2.

4.2 Impedance of a wheel. Each of the wheels in Figure 3 has the same mass, $m = 1$ [kg], and radius, $R = 1$ [m], and each sits on the same downhill slope, γ [radians]. ($g = 9.81$ [m/s^2]) However, each wheel has a different inertia. The wheel at left is a solid disk and has an inertia $J_a = \frac{1}{2}mR^2$. The center wheel is hollow, with all mass concentrated in a hoop at radius R, so $J_b = mR^2$. The wheel at right has all mass at a single point at its center, with massless spokes and rim, so $J_c = 0$.

a) Write the equation of motion for each wheel using x (the position of the center of the wheel along the ramp direction) as the degree of freedom in each case.

b) What will the velocity, $\dot{x}$, of each wheel be 10 seconds after it is released from standstill?

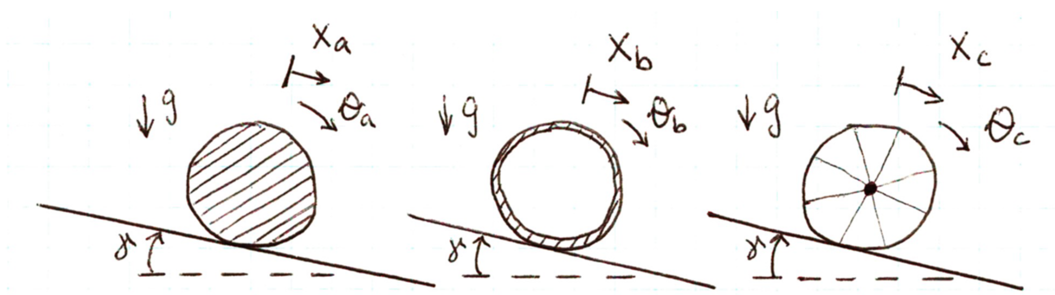

Figure 3: Three wheels, each with mass m and radius R.

4.3 Dynamics of spring-linked wheels. In this problem, we link the solid wheel ($J_a = \frac{1}{2}mR^2$) and the point-mass wheel ($J_c = 0$) from the previous problem with a spring of stiffness k. (Assume that with $x_1 = 0$ and $x_2 = 0$, the spring is at its undeflected length where it produces no force; otherwise, it is a linear spring: $F_k = k\Delta x$. Also assume the wheels roll with no slip.) Figure 4 shows the system.

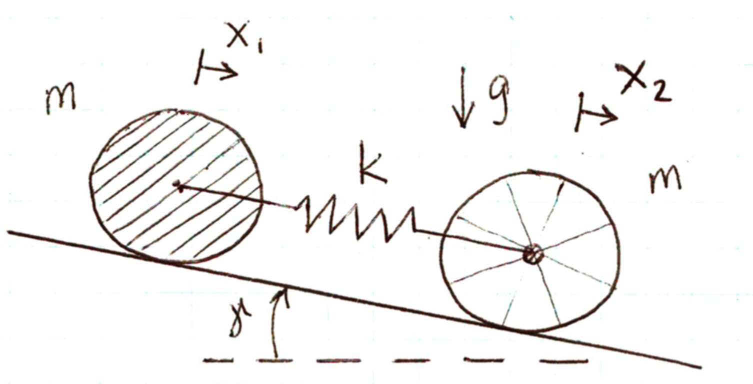

Figure 4: Solid wheel ($J = \frac{1}{2}mR^2$, at left) and point-mass wheel ($J = 0$) connected by a spring.

a) Write the equations of motion of this system, using x_1 and x_2 as the two degrees of freedom.

b) **Extra Credit** The total motion of the system can be thought of as the sum of 1) an acceleration downhill and 2) an oscillation. Namely, if the undeflected spring length when $x_1 = 0$ and $x_2 = 0$ is L, there is a location along the spring that travels with a constant acceleration, $x_{node} = x_1 + f(L + x_2)$. (If we were to paint a dot on the spring here, it would simply travel at a constant acceleration.) The motion of each mass is the sum of this constant acceleration plus an oscillation at some natural frequency, ω_n. (Each mass oscillates with the same frequency but at a different amplitude, of course.) Find expressions for:

i) The fraction of the way from x_1 to x_2, f, where the spring moves with constant acceleration.

i) The natural frequency, ω_n.

4.4 **Spring-mass-damper in series.** Figure 5 shows a system with a mass ($F = m\ddot{x}$), a spring ($F = k\Delta x$), and a damper (aka "dashpot", $F = b\Delta\dot{x}$) connected as shown. Using Laplace notation, the impedance (relating force to velocity at location i) of each can be written as:

$Z_{mass} = ms$, so $F_i = ms\dot{x}_i$

$Z_{damper} = b$, so $F_i = b(\dot{x}_i - \dot{x}_j)$

$Z_{spring} = \frac{k}{s}$, so $F_i = k(x_i - x_j)$

(Note the force at x_i due to a damper/(spring) depends on a relative velocity/(position) with respect to another location (x_j) in the system.)

b) Find the impedance relationship between F (the force) and $\dot{x}_3$ (the velocity where the force is applied):

$Z_{tot} = \frac{F}{\dot{x}_3}$.

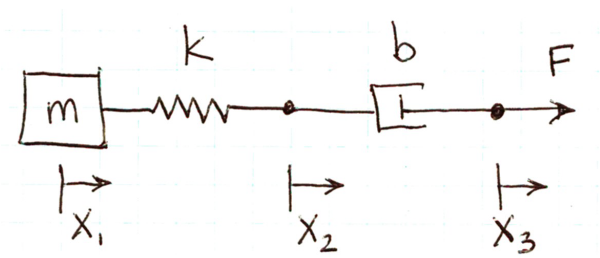

Figure 5: Spring-mass-damper system "in series".

References

[1] Mark W. Spong. Partial feedback linearization of underactuated mechanical systems. In *Proceedings of the IEEE International Conference on Intelligent Robots and Systems*, volume 1, pages 314–321, September 1994.