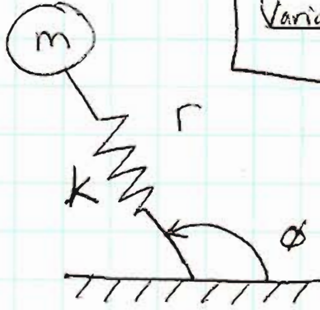
"SLIP": Lagrangian Equations of Motion Derivation

(Spring-Loaded Inverted Pendulum)

Constants → m : point mass , k: spring constant
Variables → r : leg length , ϕ : leg angle



Kinetic co-energy → $T^* = \frac{1}{2} m v^2$

Potential energy → $V = mgh + \frac{1}{2}k(l_0 - r)^2$

- Position & velocity of mass are:

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$\dot{x} = -r\sin\phi\,\dot{\phi} + \dot{r}\cos\phi \qquad \dot{y} = r\cos\phi\,\dot{\phi} + \dot{r}\sin\phi$$

$$\therefore v^2 = \dot{x}^2 + \dot{y}^2 = (-r\sin\phi\,\dot{\phi} + \dot{r}\cos\phi)^2 + (r\cos\phi\,\dot{\phi} + \dot{r}\sin\phi)^2$$

$$= \{r^2\sin^2\phi\,\dot{\phi}^2 - 2r\dot{r}\sin\phi\cos\phi\,\dot{\phi} + \dot{r}^2\cos^2\phi\} + \ldots$$

$$\{r^2\cos^2\phi\,\dot{\phi}^2 + 2r\dot{r}\cos\phi\sin\phi\,\dot{\phi} + \dot{r}^2\sin^2\phi\}$$

$$v^2 = r^2(\sin^2\phi + \cos^2\phi)\dot{\phi}^2 + \dot{r}^2(\cos^2\phi + \sin^2\phi)$$

$$\boxed{v^2 = r^2\dot{\phi}^2 + \dot{r}^2}$$

You can also just write this directly by noticing the velocity has radial & tangential magnitudes of "$\dot{r}$" and "$r\dot{\phi}$"...

- Height of mass is:

$$\boxed{h = y = r\sin\phi}$$

So,

$$\boxed{\begin{aligned} T^* &= \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2) \\ V &= mgr\sin\phi + \frac{1}{2}k(l_0 - r)^2 \end{aligned}}$$

"Lagrangian"

$$\boxed{\mathcal{L} = T^* - V = \frac{m}{2}(r^2\dot{\phi}^2 + \dot{r}^2) - mgr\sin\phi - \frac{k}{2}(l_0 - r)^2}$$

Now, for each degree of freedom (r & ϕ),

$$\tilde{\lambda}_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}$$

$\tilde{\lambda}_i$: generalized FORCE (no external forces in SLIP model)

q_i : generalized coordinate (r or ϕ)

{recall, $\mathcal{L} = \frac{m}{2}(r^2\dot{\phi}^2 + \dot{r}^2) - mgr\sin\phi - \frac{k}{2}(l_0 - r)^2$ }

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in STANCE

for r,

$$0 = \frac{d}{dt}(m\dot{r}) - (mr\dot{\phi}^2 - mg\sin\phi + k(l_0 - r))$$

$$0 = m\ddot{r} - mr\dot{\phi}^2 + mg\sin\phi - k(l_0 - r)$$

$$\boxed{\ddot{r} = r\dot{\phi}^2 - g\sin\phi + k(l_0 - r)}$$

for ϕ,

$$0 = \frac{d}{dt}(mr^2\dot{\phi}) - (-mgr\cos\phi)$$

$$0 = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} + mgr\cos\phi$$

$$\ddot{\phi} = \frac{-1}{mr^2}(2mr\dot{r}\dot{\phi} + mgr\cos\phi)$$

$$\boxed{\ddot{\phi} = \frac{-1}{r}(2\dot{r}\dot{\phi} + g\cos\phi)}$$

Note, in FLIGHT, the mass has a ballistic trajectory:

$$\boxed{\begin{aligned}\ddot{x} &= 0\\ \ddot{y} &= -g\end{aligned}}$$