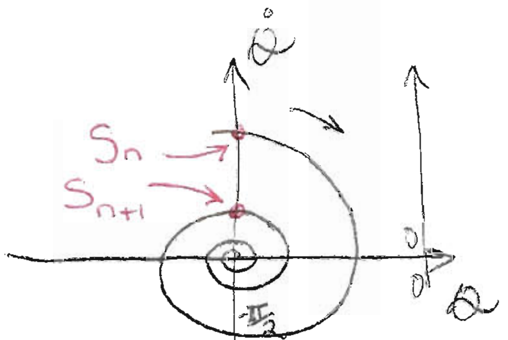
Limit Cycle Stability

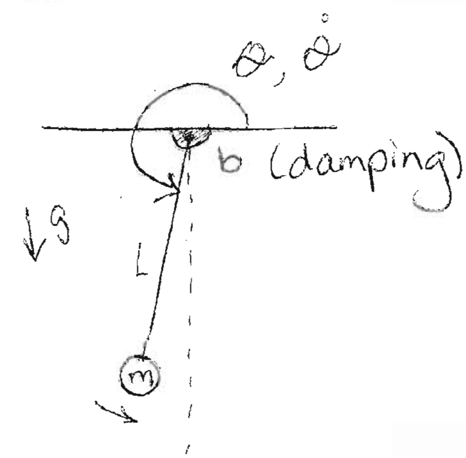
1/11/2010
ECE594D

For a cyclical motion (e.g., walking, pendulum, ...), we can look at the state each time it crosses a particular "section" (slice) in state space: (This is a "Poincaré section".)

Pendulum example



(Trajectory in Phase Space)



$$\ddot{\theta} = \frac{-g}{L}\sin\left(\theta + \frac{\pi}{2}\right) - b\dot{\theta}$$

Let "S" be our state, $S = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$.

- The <u>return map</u> is the function that tells you where S_{n+1} will "hit" on your Poincaré section, given you start w/ $S = S_n$: $S_{n+1} = F(S_n)$

"return function" or "return map"

- A fixed point is a location where $S^* = F(S^*)$, (that is, where $S_{n+1} = S_n$).

- How to determine if S^* is a stable fixed point:

For each of the n degrees of freedom,

- MOVE to a point in state space a teeny distance away from S^*, in ONLY the i^{th} DOF.

$$S^* + \Delta S_n = S^* + \begin{bmatrix} \Delta \\ 0 \end{bmatrix}$$

i^{th} element, for $i=1$

($n = 2$ DOF, total.)

- Observe next crossing of Poincaré section:

$$S_{n+1} = F(S_n) = F\left(S^* + \begin{bmatrix} \Delta \\ 0 \end{bmatrix}\right)$$

- We can rewrite S_{n+1} as:

$$S_{n+1} = S^* + \begin{bmatrix} J_{1i} \\ J_{2i} \end{bmatrix}$$

for a change in the i^{th} DOF of the input

- ...where the $n \times n$ matrix $\mathbf{J}$ is the JACOBIAN, which shows $\frac{\partial F}{\partial S}$.

Then, repeat for i equal to each of the n total DOF's, to get each column in $\mathbf{J}$.

- The elements of the <u>Jacobian</u> are:

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

Column matches OUTPUT DOF

Row corresponds to which INPUT DOF

- For the step-to-step mappings,

$$\boxed{\begin{aligned} x &= S_n \\ y &= S_{n+1} \end{aligned}}$$

- Finally, the <u>eigenvalues</u> of J tell you about <u>STABILITY</u> of/near S^*:

$$\lambda \Rightarrow \text{eigenvalues of } J$$

- For <u>stable</u> fixed points,

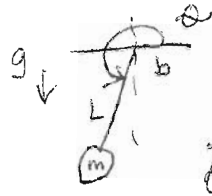
$$|\lambda_i| < 1 \quad , \forall i$$

- For <u>unstable</u> fixed points,

$$|\lambda_i| > 1 \quad , \text{for } \underline{\underline{\text{any}}} \ i$$

Jacobian example for Pendulum with Damping:

$$S^* = \begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}$$



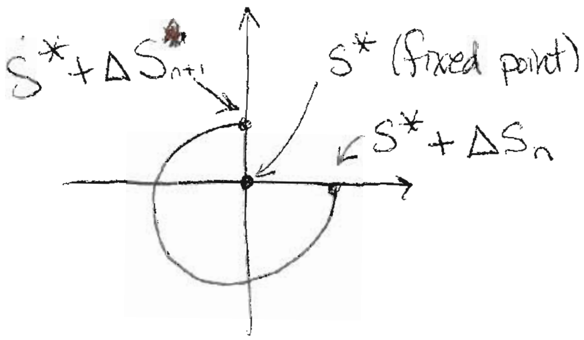
$L = 1\ (m)$

$b = 1\ (s^{-1})$

$g = 9.8\ (m/s^2)$

$$\ddot{\theta} = \frac{g}{L}\sin(\theta) - b\dot{\theta}$$

$\boxed{i = 1\ (\theta)}$, $\Delta S_n = \begin{bmatrix} d \\ 0 \end{bmatrix}$



Say $= .001$,

$$F(S^* + \Delta S_n) = F\left(\begin{bmatrix} \frac{-\pi}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} .001 \\ 0 \end{bmatrix}\right)$$

& $F(S^* + \Delta S_n) = S^* + \Delta S_{n+1}$

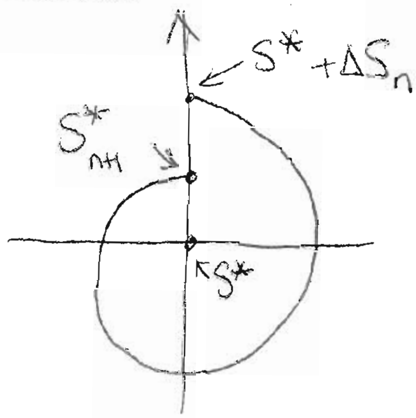
We find (numerically),

$$F\left(\begin{bmatrix} \frac{-\pi}{2} + .001 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{-\pi}{2} \\ 0.00142 \end{bmatrix}$$

$$\Delta S_{n+1} = \begin{bmatrix} \frac{-\pi}{2} \\ 0.00142 \end{bmatrix} - \underbrace{\begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}}_{S^*}$$

$$J(:,1) = \frac{\Delta S_{n+1}}{d} = \begin{bmatrix} 0 \\ 1.4188 \end{bmatrix}$$

$\boxed{i = 2\ (\dot{\theta})}$, $\Delta S_n = \begin{bmatrix} 0 \\ \Delta \end{bmatrix}$



(again, $d = .001$)

$$F(S^* + \Delta S_n) = F\left(\begin{bmatrix} \frac{-\pi}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ .001 \end{bmatrix}\right)$$

(numerical solution)

$$F\left(\begin{bmatrix} -\pi/2 \\ .001 \end{bmatrix}\right) = \begin{bmatrix} -\pi/2 \\ .0003587 \end{bmatrix}$$

$$\Delta S_{n+1} = \begin{bmatrix} \frac{-\pi}{2} \\ .0003587 \end{bmatrix} - \overbrace{\begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}}^{S^*}$$

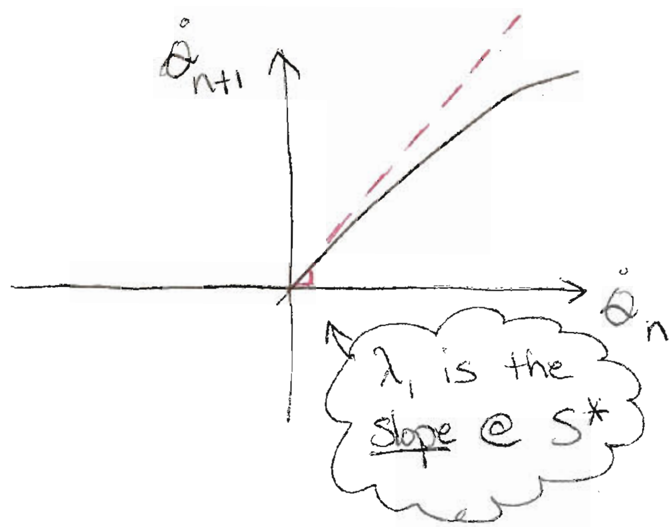
$$J(:,2) = \frac{\Delta S_{n+1}}{d} = \begin{bmatrix} 0 \\ .3587 \end{bmatrix}$$

So, $J = \begin{bmatrix} 0 & 0 \\ 1.4188 & .3587 \end{bmatrix}$

$\lambda_1 = .3587$

$\lambda_2 = 0$ $\left\{ \leftarrow \text{b/c } \theta = \frac{-\pi}{2} \text{ EXACTLY, b/c we define the Poincaré section this way.} \right\}$

So, if we have a (nonlinear) return map for $\dot{\theta}$,



(Example code will appear on HW webpage...)

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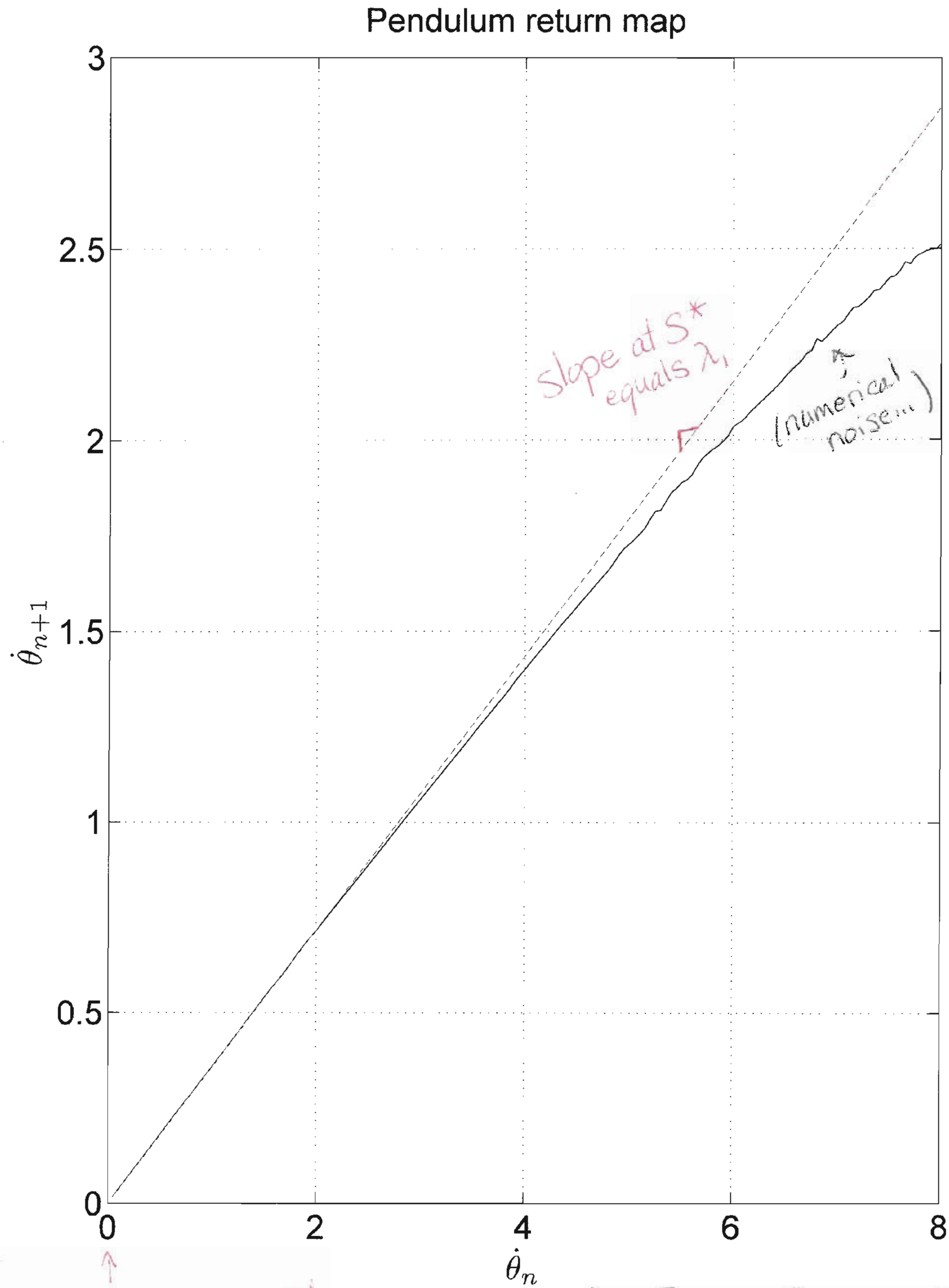

$\dot{\theta}$ at S* is $0 \frac{\text{rad}}{\text{sec}}$

($\theta = \frac{-\pi}{2}$ on the Poincaré section.)