ECE594I, Prof. Brown, Fall Quarter 2008

Homework Problems #1: Fundamentals of Radiation and Fluctuations

1. Planck's law of radiation is a triumph of quantum physics and very useful in many fields of science and engineering. (a). Given a 300 K blackbody of unity emissivity, find the frequency in THz (accurate to 100 GHz) and the wavelength in micron where the brightness form of Planck's law is a maximum (can use numerical or analytic techniques). (b) What is this maximum brightness in MKS units ? (c) what is the brightness of this same blackbody at 1.0 THz and how much less is this than the maximum ?

2. A useful function for practically any electromagnetic sensor is the *integrated* brightness [i.e., integrated over frequency]. (a) Derive an expression for the integrated brightness IB in terms of the temperature and the Stefan-Boltzman constant, $\sigma = (2\pi h/15c^2)(\pi k_B T/h)^4$. (b) Evaluate IB at T = 300 K. (c) Now consider the fractions of IB that lies above and below a certain "cutoff" frequency, $IB(>\nu c) \equiv \int_{\nu c}^{\infty} B(\nu)d\nu$ and $IB(<\nu c) \equiv \int_0^{\nu c} B(\nu)d\nu$ [both are very useful in evaluating the "background" radiation for photon and thermal detectors.]. In the THz region and for terrestrial sources, one can generally apply the Rayleigh-Jeans approximation, $h\nu c/k_B T << 1$. In this case, derive an expression for $IB(<\nu c)$, and then $IB(>\nu c)$. (d) Evaluate $IB(<\nu c)$ for νc = 1 THz and T = 300 K. [suggestion: consult Eisberg&Resnick for an interesting discussion on the genesis of Planck's law and the Stefan-Boltzman constant].

3. THz Radiation Filter: (a) Suppose one is using a thermal detector (e.g., Golay cell) that is approximately equally sensitive to IR and THz radiation. Clearly, a low-pass filter is desirable to attenuate (i.e., "stop") the IR but "pass" the THz. For the first approximation, one can assume that the low-pass filter has a power transmission function, $T = |S_{21}|^2 = \tau\cdot\theta(\nu - \nu_C) + \theta(\nu_C - \nu)$, where θ is the unit step function and τ is the "average" transmission above ν_C. (a) Assuming that the filter is operated at room temperature but does not emit any radiation towards the detector, and that ν_C satisfies the Rayleigh Jeans limit, write an expression for the value of τ that makes the incident power on the detector from $\nu > \nu c$ equal to that from $\nu < \nu c$. (b) Evaluate this τ for νc = 1 THz and T = 300, and state the answer as an insertion loss in decibel units. (c) In the stop band region what is a better type of attenuation for the filter to have: absorptive or scattering ? State your answer qualitatively assuming that the filter can be fabricated as a uniform film that completely intercepts the incoming radiation to the detector ? [clue: there are two reasons for the correct answer, one of which involves Kirchoff's law of radiative transfer].

4. Fluctuations of Radiation: the quantum picture. Given the quantum statistics for radiation based on Boltzman and Planck, it is simple to derive the fluctuations from general principles of probability theory.

(a) Derive the variance or mean-square fluctuations, $<(\Delta n_K)^2>$ for the radiation modeled as photons having the energy function $U_K = (n_K + ½)h\omega_K$, where n_K is the number of photons in mode K. [clue: start with the Boltzman (exponential) PDF and utilize the general result for random variables, $<(\Delta n_K)^2> = <(n - <n_K>)^2> = <(n_K)^2> - <n_K>^2$].

(b) Use the above expression to calculate the root mean square fluctuations of incident power in a single spatial mode from the sun within a spectral bandwidth of 1 GHz at the following two center wavelengths: (1) λ = 0.5 mm (THz region), and (2) λ = 0.5 micron (visible region). Express your answer in MKSA and in dBm (decibels relative to 1 mW).

[clues: make the narrow passband approximation in both cases; approximate the sun as having a brightness temperature of 5800 K].

5. Classical Picture of Radiation Fluctuations: Gaussian noise. The Johnson-Nyquist theorem predicts an available RMS noise power of $k_B T\Delta\nu$ per spatial mode, where k_B is Boltzman's constant and $\Delta\nu$ is the instantaneous bandwidth. This can be derived from purely classical principles if we ignore the quantization of radiation modes (i..e., photons) and deal with the continuous Boltzman distribution function $p(U_K) = C\exp(-U_K/k_B T)$ where U_K is the energy contained in the electromagnetic wave of wavevector K. To make the analysis simpler, we assume the radiation is a unidirectional TEM mode on a transmission line.

a. By integrating over U_K as a continuous variable, find the constant C so that this distribution is properly normalized.

b. Find the mean energy as a function of U_K .

c. Relate the average power on the transmission line to the average power through $P_K = cU_K/L = BU_K$ where L is the length of line and B is the bandwidth. Use this to find the power distribution function $p(P_K) = D\exp(-EP_K/k_B T)$. Use the normalization condition and definition of ensemble average $\langle P\rangle$ to find the constants D and E in terms of $\langle P\rangle$ and B.

d. Use this new function $p(P_K)$ to find the power variance $\langle(\Delta P_K)^2\rangle$ in terms of $\langle P_K\rangle$

e. Now use the special relationship for TEM modes, $P_K = (V_K)^2/Z_0$, for the instantaneous power to re-write $p(P_K)$ in terms of V_K and $\langle(\Delta V_K)^2\rangle$, assuming that $\langle V_K\rangle = 0$. This is the familiar Gaussian distribution function for the randomly fluctuating voltage from thermal noise, and is extremely useful in circuit and systems theory.

6. Poisson distribution and oscillator fluctuations.
The Poisson probability distribution $P(n,\langle n\rangle) = \langle n\rangle^n \exp(-\langle n\rangle)/n!$. It defines the probability of arrival of n photons to a sensor during any time interval in which the mean number (made over many different measurements over the same interval) is $\langle n\rangle$. It is a rigorous description of the quantum-limited noise of any "coherent state", which corresponds to a sinusoidal oscillator in the classical limit.

a. Show that $P(n,\langle n\rangle)$ has the required normalization property of any PDF [Clue: don't forget the general definition of a Maclaurin series].

b. Evaluate the variance or mean-square fluctuation [Clue: consider the difference $\langle n^2\rangle - \langle n\rangle$]

c. Evaluate this variance for a very common source in RF systems: a local oscillator producing an average power of 1 mW into a receiver circuit having a 10% instantaneous bandwidth.