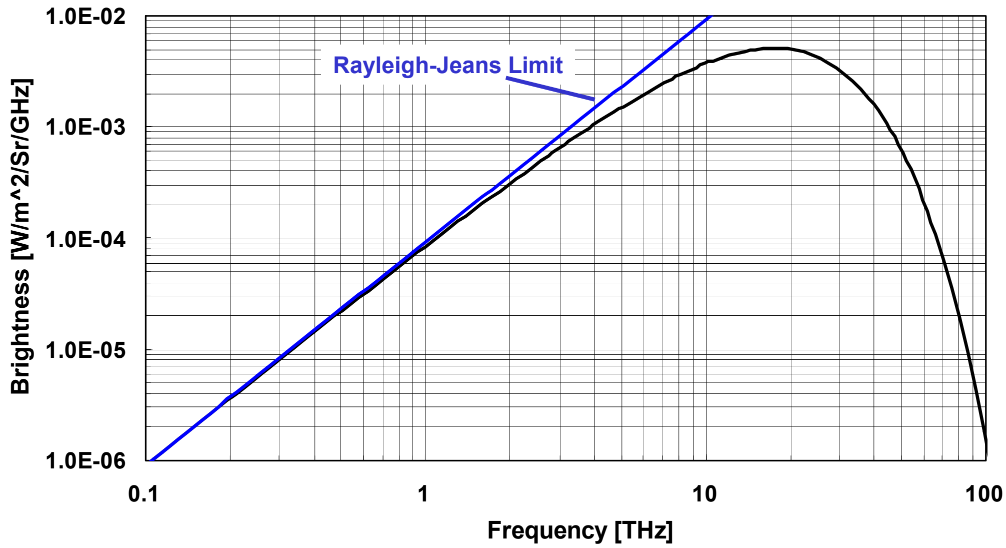
ECE594I, Prof. Brown, Fall Quarter 2008

HW Solutions #1



Brightness function for Problem 1 (unity emissivity and T = 300 K).

1.(a) From the course lecture notes, the brightness function is given by $\frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/k_BT}-1}$ [W/m^2/Sr]. A plot of this brightness function vs frequency is shown below for T = 300 K. (a) by inspection and iteration, it is easy to find that the maximum value of B occurs at 17.6 THz (λ = 17.0 μm). (b) The maximum brightness function is $5.105\text{x}10^{-12}$ W/(m^2-Hz). (c) The brightness function at ν = 1.0 THz is $8.48\text{x}10^{-14}$ W/(m^2-Hz), 60.2 times lower than the maximum !

2. The integrated brightness function is defined and evaluated as follows:

$$IB = \int_0^\infty \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/k_BT}-1} = \frac{2h}{c^2}\left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{x^3dx}{e^x-1} = \frac{2h}{c^2}\left(\frac{k_BT}{h}\right)^4 \frac{\pi^4}{15} \equiv \sigma T^4/\pi$$

where $\sigma = 5.67\text{x}10^{-8}$ W/(m^2-K^4) [note the good mnemonic ordering of numbers] (b) At T = 300 K, IB = 146.2 W/(m^2-Sr). (c) In the (long-wavelength) Rayleigh-Jeans limit, $B_T = 2k_BT(\nu/c)^2$, so that

$IB(<\nu_C) = \int_0^{\nu_C} \frac{2k_BT\nu^2}{c^2} d\nu = 2k_BT(\nu_C)^3/3c^2$. And using elementary calculus, we find

$IB(>\nu_C) = \int_{\nu_C}^\infty B(\nu)d\nu = \int_0^\infty B(\nu)d\nu - \int_0^{\nu_C} B(\nu)d\nu = \sigma T^4/\pi - 2k_BT(\nu_C)^3/3c^2$. (d) For ν_C = 100 GHz, IB ($<\nu_C$) = 0.031 mW/(m^2-Sr), which is a tiny fraction $2.1\text{x}10^{-7}$ of the total IB.

3. (a) Variance for photons

$\langle(\Delta n_K)^2\rangle = \langle(n-\langle n_K\rangle))^2\rangle = \langle(n_K)^2\rangle - \langle n_K\rangle^2$, where $\langle n_K\rangle = [\exp(h\nu/k_BT)-1]^{-1}$ is the Planck function as derived from the Boltzman (exponential) density function

$$< n_K >= \frac{\sum_{n_K=0}^{\infty} n_K \exp[-n_K h\nu / k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu / k_B T]} \text{ and } < (n_K)^2 >= \frac{\sum_{n_K=0}^{\infty} n_K^2 \exp[-n_K h\nu / k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu / k_B T]}$$

This is best evaluated with the trick identity

$n_K^2 = \left(\frac{k_B T}{h}\right)^2 \left(\frac{d^2}{d\nu^2} \exp(-n_K h\nu / k_B T\right) \cdot \exp(nh\nu / k_B T)$, so the numerator above becomes

$$< (n_K)^2 >= (k_B T / h)^2 \sum_{n_K=0}^{\infty} (d^2 / d\nu^2) \exp(-n_K h\nu / k_B T) = (k_B T / h)^2 (d^2 / d\nu^2) \sum_{n_K=0}^{\infty} \exp(-n_K h\nu / k_B T)$$

$$= (k_B T / h)^2 (d^2 / d\nu^2) \left[\frac{1}{1 - \exp(-h\nu / k_B T)}\right] = \frac{\exp(-h\nu / k_B T)[1 + \exp(-h\nu / k_B T)]}{[1 - \exp(-h\nu / k_B T)]^3}$$

Thus, we have $< (n_K)^2 >= \dfrac{\sum_{n_K=0}^{\infty} n_K^2 \exp[-n_K h\nu / k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu / k_B T]} = \dfrac{\dfrac{\exp(-h\nu / k_B T)[1 + \exp(-h\nu / k_B T)]}{[1 - \exp(-h\nu / k_B T)]^3}}{\dfrac{1}{1 - \exp(-h\nu / k_B T)}}$

$$< (n_K)^2 >= \frac{\exp(-h\nu / k_B T)[1 + \exp(-h\nu / k_B T)]}{[1 - \exp(-h\nu / k_B T)]^2} = \frac{\exp(h\nu / k_B T)[1 + \exp(-h\nu / k_B T)]}{[\exp(h\nu / k_B T) - 1]^2} = \frac{\exp(h\nu / k_B T) + 1)}{[\exp(h\nu / k_B T) - 1]^2}$$

$$<(n_K)^2> - <n_K>^2 = \frac{\exp(h\nu / k_B T) + 1}{[\exp(h\nu / k_B T) - 1]^2} - \frac{1}{[\exp(h\nu / k_B T) - 1]^2} = \frac{\exp(h\nu / k_B T)}{[\exp(h\nu / k_B T) - 1]^2}$$

$=< n_K >^2 (< n_K >^{-1} + 1) =< n_K > (< n_K > +1)$ {The famous "photon bunching" expression}

(b) For the sun radiation at T_B = 5800 K in a single spatial mode:

- at λ = 0.5 mm, ν = 0.6 THz, $<n_K>$ = 200.8, $<(\Delta n_K)^2>$ = 4.05×10^4

$<P_K> = <n_K>h\nu\Delta\nu = 8.0 \times 10^{-11}$ W ; $<(\Delta P_K)^2> = <(\Delta n_K)^2>(h\nu\Delta\nu)^2 = 6.4 \times 10^{-21}$ W
so $(P_K)_{rms}$ = 8.0×10^{-11} W = -71 dBm

- at λ = 0.5 micron, $\nu = 6 \times 10^{14}$ Hz, $<n_K> = 7.0 \times 10^{-3}$, $<(\Delta n_K)^2> \approx 7.0 \times 10^{-3}$

$<P_K> = <n_K>h\nu\Delta\nu = 2.8 \times 10^{-12}$ W; $<(\Delta P_K)^2> = <(\Delta n_K)^2>(h\nu\Delta\nu)^2 = 1.1 \times 10^{-21}$ W
so $(P_K)_{rms}$ = 3.3×10^{-11} W = -75 dBm.

4. For the Boltzman (or exponential) probability density function, $p(U_K) = C\exp(-U_K/k_BT)$

(a) Normalization => $C \cdot \int_0^{\infty} \exp(-U_K / k_B T) dU_K = 1 = -Ck_B T \cdot \exp(-U_K / k_B T) \Big|_0^{\infty} = Ck_B T$

so that $C = 1/(k_BT)$.

(b) $<U_K> = (k_BT)^{-1}\cdot\int_0^\infty U_K\exp(-U_K/k_BT)dU_K$ which we integrate "by parts", using $V = U_K$ and $dW = \exp(-U_K/k_BT)dU_K$. So we get $<U_K>= VW\big|_0^\infty - \int_0^\infty WdV =$

$$(k_BT)^{-1}\left[(k_BT)^{-1}U_K\exp(-U_K/k_BT)\Big|_0^\infty + (k_BT)\int_0^\infty \exp(-U_K/k_BT)dU_K\right] = k_BT$$

(c) In terms of the average power P_K, $U_K = P_K/B$ where B is the (longitudinal-mode) bandwidth. So if we define $p(P_K) = D\exp(-EP_K/k_BT)$, then by inspection, $E = 1/B$ and $dU_K = dP_K/B$. And the normalization condition =>

$$D\cdot\int_0^\infty \exp(-P_K/Bk_BT)dP_K = 1 = D\cdot[-Bk_BT\cdot\exp(-P_K/k_BTB)]\big|_0^\infty = DBk_BT$$ so that $D = (k_BTB)^{-1}$.

(d) Power variance $<(\Delta P_K)^2> = <(P_K)^2> - <P_K>^2$. The first term is evaluated with the substitution $x = P_K/(Bk_BT)$. $<(P_K)^2> =$

$$(k_BTB)^{-1}\cdot\int_0^\infty P_K^2\exp(-P_K/Bk_BT)dP_K = (k_BTB)^{+2}\int_0^\infty x^2\exp(-x)dx = (k_BTB)^{+2}\cdot 2$$ where the last step follows from consulting any table of definite integrals. Thus, we arrive at

$<(\Delta P_K)^2> = 2(k_BTB)^2 - <P_K>^2 = 2(k_BTB)^2 - (k_BTB)^2 = (k_BTB)^2$.

(e) Gaussian form. *First note that from the above derivations, we can write:* $p(P_K) = (<P_K>)^{-1}\exp(-P_K/<P_K>)$. From transmission line (and TEM-mode) theory, we know the time average power $(P_K)_{ave} = |\mathbf{V}_K|^2/2Z_0$ and the instantaneous power is $P_K= (V_K)^2/Z_0$ where $\mathbf{V}_K$ is the phasor voltage and V_K is the (real) instantaneous voltage. So if P_K is considered as a random variable fluctuating between 0 and infinity, V_K must also be a random variable varying between –infinity and infinity. Hence, we can transform $p(P_K)$ to $p(V_K)$ by $p(V_K) = F\exp[-G(V_K)^2/(Z_0<P_K>)]$ where F and G are determined by the normalization condition and by requiring that $F\int_{-\infty}^{\infty} P_K\exp[-G(V_K)^2/(Z_0<P_K>)]dV_K = <P_K>$. After use of the Gaussian integrals {e.g., $\int_{-\infty}^{\infty}\exp(-ax^2)dx = \sqrt{\pi/a}$}, one finds

$p(V_K) = [2\pi<(\Delta V_K)^2>]^{-1/2}\exp[-(V_K)^2/(2<(\Delta V_K)^2>)]$ assuming that $<V_K> = 0$.

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This is the classic Gaussian distribution function for a fluctuating voltage in any transmission-line mode, and also applies to any circuit whose length is small compared to a free-space wavelength.

5. Poisson distribution: $P(n,<n>) = <n>^n\exp(-<n>)/n!$

(a). Normalization property requires consideration of sum: $\sum_{n=0}^{\infty} P(n,<n>) = C$ where C is the normalization constant. Recall definition of Maclaurin series for a continuous function f(x) close to x =0:

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{df}{dx} \right)^n \Bigg|_{x=0} (x)^n$ where 0! = 1 and $(df/dx)^0 = f(x=0)$ by definition . From this we recognize that the natural exponent has the Maclaurin series $\exp(x) = 1 + x + x^2/2 + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By associating x and <n>, we get

$$\sum_{n=0}^{\infty} <n>^n \exp(-<n>)/n! = \exp(-<n>) \cdot \sum_{n=0}^{\infty} <n>^n \; /n! = \exp(-<n>) \cdot \exp(<n>) = 1$$

so that C = 1 and the Poisson PDF is properly normalized.

(b).Variance: $<\Delta n)^2 = <(n-<n>)^2> = <n^2> - <n>^2$. We start by considering the trick:

$$<n^2> - <n> = <n^2 - n> = <n(n-1)> = \sum_{n=2}^{\infty} n(n-1) \cdot <\mathrm{n}>^n \exp(-<\mathrm{n}>)/[\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)!]$$ where

the first two terms in the sum (n = 0 and n = 1) have vanished because of the n(n-1) factor. Thus, the sum can start at n = 2, and because of cancellation of n(n-1) we get

$$<n^2> - <n> = \sum_{n=2}^{\infty} <\mathrm{n}>^n \exp(-<\mathrm{n}>)/[(\mathrm{n}-2)!] = \sum_{m=0}^{\infty} <\mathrm{n}>^{m+2} \exp(-<\mathrm{n}>)/[(\mathrm{m})!]$$

where we have substituted m = n-2. Finally, extracting an $<n>^2$ factor yields

$$<n^2> - <n> = <n>^2 \sum_{m=0}^{\infty} <n>^m \exp(-<n>)/[(m)!] = <n>^2$$

since the last sum is just the normalized Poisson PDF with a different (arbitrary) summation index. Thus, we have $<n^2> - <n> = <n>^2$, or $<n^2> - <n>^2 = <(\Delta n)^2> = <n>$.

(c) An RF local oscillator will produce a mean number of photons that always depends on the measurement, or sampling time t_s. According to quantum theory, the photon rate is just $P_{ave}/h\nu_0$ where P_{ave} is the average power and ν_0 is the oscillator center frequency. So the number of photons measured is $n_{ave} = P_{ave} t_s/h\nu_0$. And according to sampling theory, $t_s = 1/\Delta\nu$, where $\Delta\nu$ is the instantaneous bandwidth of the external circuit (note: any good oscillator will have a "linewidth" $\delta\nu$ much less than $\Delta\nu$). So we get $n_{ave} = P_{ave} /(h\nu_0 \Delta\nu) \equiv <n>$, where we have equated a time average n_{ave} to an ensemble average <n> (the important principle in probability theory called the "ergodic" hypothesis). So according to the variance result derived above, $<(\Delta n)^2> = <n> \equiv n_{ave} = P_{ave} /(h\nu_0 \Delta\nu)$. As an example, if the bandwidth is 10% and the average power is 1 mW, we have $\Delta\nu/\nu_0 = 0.1$ and we get $<n> = 0.01/[h(\nu_0)^2]$. Furthermore, if we take ν_0 = 10 GHz (a common RF sensor frequency), we get, $<(\Delta n)^2> = 1.5\mathrm{x}10^{11}$ or $n_{rms} = 3.88\mathrm{x}10^5$. This surprisingly large rms fluctuation is a consequence of the sampling time. When one samples longer, more photons are measured and, therefore, more fluctuation occurs.