Radiation Fluctuations

- Quantum mechanics teaches us that the act of measuring any physical observable is inherently probabilistic with a metric of uncertainty given by Planck's constant h.
- Expect fluctuation in power from any radiative source, even coherent ones.
- Noise theory is relatively simple in two extremes: (1) coherent radiation, or (2) thermal radiation

Consider n, the number of photons in spatial mode m, as a random variable:

$$\Delta P_{inc} \equiv P_{inc} - < P_{inc} > = \sum_{m}^{M} \int_{\nu_0}^{\nu_0+\Delta\nu} h\nu \cdot \{n_m(\nu) - < n_m(\nu) >\} d\nu \equiv \sum_{m}^{M} \int_{\nu_0}^{\nu_0+\Delta\nu} h\nu \cdot \Delta n_m(\nu) \cdot d\nu$$

Important measure of noise is the variance, or mean-square fluctuation

$$< (\Delta P_{inc})^2 > = < (\sum_{m}^{M} \int_{\nu_0}^{\nu_0+\Delta\nu} h\nu \cdot \Delta n_m(\nu) \cdot d\nu) \cdot (\sum_{m'}^{M'} \int_{\nu_0}^{\nu_0+\Delta\nu} h\nu \cdot \Delta n_{m'}(\nu) \cdot d\nu) >$$

where m' is a dummy summation index. In evaluating this expression we utilize the fact that perfectly *random* fluctuations in different orthogonal modes are uncorrelated, so that only cross-products having the same mode index will survive the ensemble average

$$< (\Delta P_{inc})^2 > = < \sum_{m}^{M} (\int_{\nu_0}^{\nu_0+\Delta\nu} h\nu \cdot \Delta n_m(\nu) \cdot d\nu)^2 >$$

This is greatly simplified by the "band-limited"`assumption: $\Delta\nu$ is so narrow that both $h\nu$ and $\Delta\nu$ can be considered constant over the range of the integration.

$$< (\Delta P_{inc})^2 > \approx < \sum_{m}^{M} (h\nu_0 \Delta n_m(\nu_0) \cdot \Delta\nu)^2 > = \sum_{m}^{M} \cdot (h\nu_0 \Delta\nu)^2 < [\Delta n_m(\nu_0)]^2 >$$

$$< (\Delta P_{abs})^2 > \approx < \sum_{m}^{M} \eta_m \cdot (h\nu_0 \Delta n_m(\nu_0) \cdot \Delta\nu)^2 > = \sum_{m}^{M} \eta_m \cdot (h\nu_0 \Delta\nu)^2 < [\Delta n_m(\nu_0)]^2 >$$

In the special but common case in RF sensor of only one spatial mode, we find

$$< (\Delta P_{abs})^2 > = \eta \cdot (h\nu_0 \Delta\nu)^2 < \Delta n(\nu_0) >^2$$

Thermal Radiation

For any remote sensor calculations, particularly in the THz region, one needs to consider the fluctuations of thermal radiation. The terrestrial thermal radiation level is quite high, and even in outer space there is a (cosmic) background radiation corresponding to a 2.7-K thermal source. It is a basic exercise of statistical mechanics to show using the harmonic oscillator energy function and Boltzman statistics that

$$<(\Delta n)^2> \equiv <(n-<n>)^2> = <n^2> - (<n>)^2 = <n>(<n>+1) = f_P(1+f_P)$$

This expression has two interesting limits. In the low-frequency, or Rayleigh-Jeans limit when $h\nu << kT$, $f_P >> 1$ and $<(\Delta n)^2>$ goes to f_P^2, which is approximately $(kT/h\nu)^2$.
In the high-frequency limit when $h\nu > kT$, $f_P << 1$ and $<(\Delta \nu)^2>$ goes to $<n>$ or f_P , which goes to $e^{-h\nu/kT}$. The first limit is always valid in the RF region at terrestrial temperatures. The second limit is generally anywhere in the near-infrared or shorter wavelengths under the same terrestrial conditions. For example, for T = 300 K, $<(\Delta n)^2>$ aproaches $<n>$ asymptotically just above 10 THz

Substitution into the band-limited expression for absorbed power fluctuation of M spatial modes then yields

$$<(\Delta P_{abs})^2> \approx \sum_m^M \eta_m(\nu_0)\cdot(h\nu_0\Delta\nu)^2 <n_m>(<n_m>+1) = [\sum_m^M \eta_m(\nu_0)]\cdot(h\nu_0\Delta\nu)^2 f_P(f_P+1)$$

This expression represents a remarkable effect of thermal photons called “bunching”:
As the occupancy of photon states grows past unity, there is a tendency for the fluctuations to grow in the amplitude much more quickly . This is a ramification of the fact that photons are really massless bosons and, as such, are correlated quantum mechanically such that hey tend to condense in a single state at a rate that grows with the population of that state.

Remarkably, such “bunching” occurs routinely in THz sensors and RF sensors !

Johnson-Nyquist Theorem

A very important result for power fluctuations in the low-frequency limit. Since in this limit $h\nu/kT$ is very small, $e^{h\nu/kT} \approx 1 + h\nu/kT$ (by Taylor expansion) so that $f_P \approx kT/h\nu >> 1$, and thus $<(\Delta n)^2> \approx (kT/h\nu)^2$, which is a big number at RF frequencies for T near room temperature. Let's suppose that a sensor is receiving thermal noise power in this limit contained in M spatial modes over an arbitrary spectral bandwidth $\Delta\nu$. The mean-square absorbed power from the statistical fluctuations Δn is given by

$$< (\Delta P_{abs})^2 > = [\sum_m^M \eta_m](h\nu_0 \Delta\nu)^2 f_P (f_P + 1) \approx \sum_m^M \eta_m (h\nu_0 \Delta\nu)^2 (kT / h\nu_0)^2 = [\sum_m^M \eta_m](kT\Delta\nu)^2$$

$$P_{RMS} \equiv \sqrt{< (\Delta P_{abs})^2 >} = \sqrt{\sum_m^M \eta_m} \cdot k_B T \cdot \Delta\nu = < P_{abs} > \Big/ \sqrt{\sum_m^M \eta_m}$$

We observe the remarkable result that for one spatial mode (M=1) and perfect coupling, the rms absorbed power fluctuation is exactly equal to the average power – a sign of Gaussian statistics !

An important consequence of the Johnson-Nyquist theorem follows from the relationship established above between the fluctuations and the power spectral density, which allows us to write

$$S_P(\nu) \approx \frac{P_{RMS}}{\Delta\nu} = \sqrt{\sum_m^M \eta_m} \cdot k_B T$$

a result of fundamental importance to RF sensor theory, and RF systems in general. Found in nearly every calculation of system-level performance and known as the Johnson-Nyquist theorem

High-Frequency Limit: Photon Shot Noise

Although RF systems are traditionally analyzed assuming validity of the Johnson-Nyquist theorem, the THz region is exceptional because for some systems hν can approach kT so that <n> drops to order unity. As done above, let's suppose that a sensor is receiving thermal noise power in M orthogonal spatial modes over a "band-limited" Δν that is narrow enough that <n> can be treated as a constant. Let's also assume that the electromagnetic waves associated with these spatial modes are all equally-well matched to the sensor (i.e., all incident waves experience the same coupling efficiency. With these assumptions, the power from the fluctuations Δn is given by

$$< (\Delta P_{abs})^2 >= \sum_m^M \eta_m \cdot (h\nu_0 \Delta\nu)^2 f_P (f_P + 1)$$

$$S_P(\nu) \approx \frac{P_{RMS}}{\Delta\nu} = h\nu_0 \sqrt{\sum_m^M \eta_m} \sqrt{f_P (f_P + 1)}$$

This can be clarified somewhat by noting that the mean power incident from all spatial modes is simply $\sum_m^M f_P \cdot h\nu \cdot \Delta\nu$ So that

$$S_P(\nu) \approx \frac{P_{RMS}}{\Delta\nu} = \frac{\sqrt{h\nu_0 \cdot [\sum_m^M \eta_m] \cdot P_{inc} \cdot (f_P + 1)}}{\sqrt{\Delta\nu}}$$

In the high-frequency limit, hν >> k_BT, we have f_P << 1 so that the fluctuations in absorbed power become

$$< (\Delta P_{abs})^2 >= \sum_m^M \eta_m \cdot h\nu_0 \cdot \Delta\nu \cdot P_{inc}$$

$$S_P(\nu) \approx \frac{P_{RMS}}{\Delta\nu} = \frac{\sqrt{h\nu_0 \cdot P_{inc} \cdot \sum_m^M \eta_m}}{\sqrt{\Delta\nu}}$$

Coherent Radiation

The opposite extreme is coherent radiation – so important to the wide array of active sensors (e.g., radars) used in the RF region, and passive coherent sensors that use a local oscillator. This state corresponds to only one electromagnetic mode at frequency ν_c which generally has a very narrow bandwidth $\delta\nu$ that is assumed to lie within the sensor passband, $\Delta\nu$. To find the noise we need to calculate the variance of the Poisson distribution – a standard exercise in probability theory. It is found that

$$< (\Delta n)^2 >= \sum_0^\infty (\Delta n)^2 p(n) =< n >$$

In other words, the variance is equal to the mean. Substitution then yields the power variance

$$< (\Delta P_{abs})^2 >\approx \sum_m^M \eta_m \cdot (h\nu_C \Delta\nu)^2 < n_m(\nu_C) >$$

To simplify this further, we note that it assumes the mean population per longitudinal mode is flat with n, which is clearly not the case with a coherent source. But we can replace the coherent source with an effective source, flat across $\Delta\nu$, through the relation $<n_m>_{eff} = P_{inc,m}/(h\nu_c \Delta\nu)$, where $P_{inc,m}$ is the time-averaged incident power in the mth spatial mode. When integrated over the sensor spectral passband, this always gives the correct photon flux, and leads to the power variance.

$$< (\Delta P_{abs})^2 >\approx \sum_m^M \eta_m \cdot h\nu_C \cdot \Delta\nu \cdot P_{inc,m}$$

In the special but practical case that the coherent power is contained in only one spatial mode, we can immediately derive the standard deviation and RF power spectral density

$$S_P(\nu) \equiv \frac{\sqrt{< (\Delta P_{abs})^2 >}}{\Delta\nu} = \frac{\sqrt{\eta \cdot h\nu_C \cdot P_{inc}}}{\sqrt{\Delta\nu}}$$

Free Space Attenuation and Radiative Transfer

A key issue with RF sensors is *attenuation,* which has two primary contributions

(1) Absorption – the irreversible conversion of electromagnetic power into heat

(2) Scattering - the reflection of radiation from dielectric or shape mismatch

In the RF region the intermolecular separation is always much less than a wavelength, so one can apply the Lambert-Beer law

$$\tau \equiv \frac{I_t}{I_i} = \exp(-\rho \cdot \sigma_A \cdot z)$$

where I_i is the incident intensity, I_t is the transmitted intensity, ρ is the particle density, σ_A is the *absorption* cross section, and z is the path length. Usually, the argument of the exponential is re-expressed as an absorption coefficient

$$\alpha = \rho \cdot \sigma_A$$

At the higher RF frequencies (mm-wave and THz regions), the shape and size of particles can lead to electromagnetic resonances. At or near resonance the particles will act like small antennas, reflecting incident radiation in the backward direction. The degree of backscatter is measured by the radar cross section

For example, a small conducting sphere, such as a raindrop, has a maximum RCS value of

$$\sigma_S \approx \pi a^2$$

where a is the radius

This is called a Mie resonance, and it occurs for metallic particles or particles of high enough dielectric constant or magnetic permeability to support strong surface currents around the sphere.

For a/λ < 1, .the RCS is much weaker and displays *Rayleigh* scattering

$$\sigma_S \approx (a/\lambda)^4$$

The effect of attenuation on sensor incident power depends on the type of sensor:

(1) Active Sensors

$$S_R(R) = \tau(\nu,R)\, S_T = \exp(-\alpha R) S_T$$

Where R is the "range" between sensor and object, S_T is the magnitude of the Poynting vector for the "active" radiation in the direction of the target, and S_R is The magnitude of Poynting vector for active radiation back at the sensor

(2) Passive Sensors

Here it is best to describe the radiative transfer in terms of the brightness function

$$B_R = \tau(\nu,R)\, B_S + (1-\tau_A)\, B_M$$

where B_R is the received brightness, B_S is the source brightness function, B_M is the "medium" brightness function and τ_A is the transmission in the presence of absorption alone. In other words, if the medium has significant absorption, then it must also have significant re-emission according to the principle of detailed balance.

Note that in the special case that the object and medium are at the same temperature AND all the attenuation is absorptive, then we have $B_R = B_S$ and

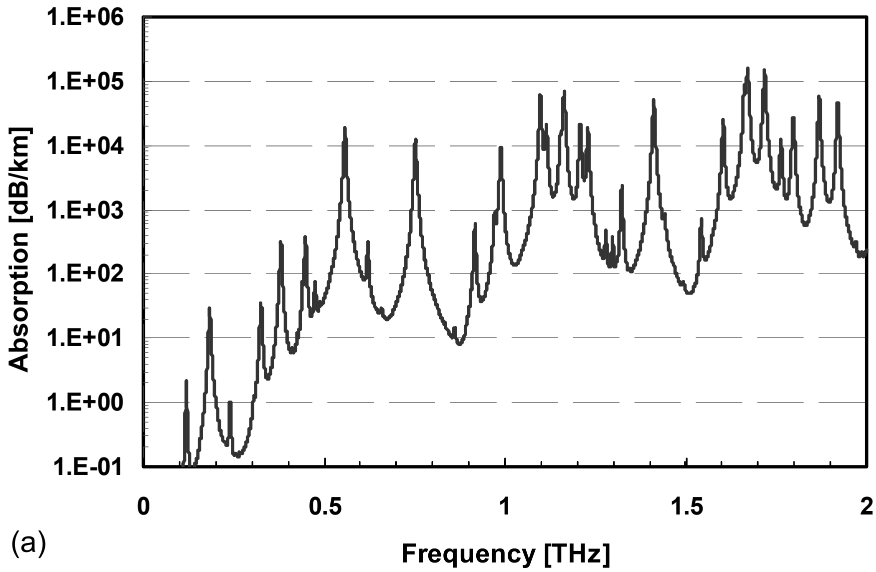
$$B_R = B_S$$

Water is the strongest atmospheric attenuator:

(1) Raindrops at frequencies of X band and above
(2) Water vapor at frequencies above ~100 GHz

Transmission of atmosphere through 1 KM of path as computed by PcLnWin:.

Assumed Conditions:
60% relative humidity, 1 atm pressure



(a)

Stick diagram of water vapor absorption lines between 0.1 and 1.0 THz (from HITRAN96 database)